Fourier (1768 - 1830)

Heine (1821 - 1881)

Dirichlet (1805 - 1859)

Dedekind (1831 - 1916)

Riemann (1826 - 1866)
Kronecker (1823 - 1891)
Hermite (1822 - 1901)
Cantor (1845 - 1918)

Schwarz (1843 - 1921)

Borel (1871 - 1956)

Baire (1874 - 1932)

Lebesgue (1875 - 1941)

Banach (1892 - 1945)

Tarski (1901 - 1983)

Fréchet (1878 - 1973)

F. Riesz (1880 - 1956)

Weyl (1885 - 1955)

Lindeberg (1876 - 1932)

Hausdorff (1868 - 1942)

Hilbert (1862 - 1943)

Poincaré (1854 - 1912)

Klein (1849 - 1925)

Kolmogorov (1903 - 1987)

Hardy (1877 - 1947)

Radon (1887 - 1956)

Gödel (1906 - 1982)

Gauss (1777 - 1855)

Weierstrass (1815 - 1897)

Bismarck (1815 - 1898, 1871)
Wilhelm II (1859 - 1941)
Hitler (1889 - 1945)
Stalin (1879 - 1953)

von Neumann (1903 - 1957)

Wiener (1894 - 1964)

Carathéodory (1873 - 1950)

Cantor (1845–1918)

Saint-Pétersbourg (étude Zürich, Berlin, Weierstrass)

Halle en 1869, 1890 fondateur et premier président DMV, ICM Zürich 1897

dès 1884 dépressions, mort clinique psychiatrique (de Halle)

Main thms: (on cardinal nos.)

- $\mathbb{Q}$, algebraic nos. are denumerable
- D den. $\Rightarrow$ $D \times D \times \dots \times D$ denumerable
- $D_1, D_2, \dots$ den. $\Rightarrow$ $\bigcup_{n=1}^{\infty} D_n$ den.
- $\mathbb{R}$ not denumerable $\quad \mathbb{R} \times \dots \times \mathbb{R} \cong \mathbb{R}$,
- if A any set then $\operatorname{card} A < \operatorname{card} \mathcal{P}(A)$
  - diagonal process.

notion of accumulation points, derived sets, closed sets

Schröder-Bernstein 1897 (Cantor 1883): $\operatorname{Card} A \leq \operatorname{card} B$, $\operatorname{card} B \leq \operatorname{Card} A \Rightarrow \operatorname{Card} A = \operatorname{Card} B$

Important papers on set theory (earlier papers 1874, 1877 $\mathbb{R}$ non den. Card hyp.)

→ Über unendliche lineare Punktmannigfaltigkeiten I–VI (Math. Ann. 1879, 80, 82, 83, 83, 84)

→ Beiträge zur Begründung der transfiniten Mengenlehre I–II (Math. Ann 1895, 1897)

transcendental nos. Liouville (1851), Hermite (e, 1873), Lindemann (π, 1882)

Gelfond-Schneider (α^β, $\alpha \neq 0, 1$, α, β algeb, $\beta \notin \mathbb{Q}$) (1934–35)

Liouville no. $\xi \in \mathbb{R}$ if
$\forall m \in \mathbb{N}^* \; \exists \frac{h_m}{k_m} \in \mathbb{Q}, k_m > 1, \left|\xi - \frac{h_m}{k_m}\right| < \frac{1}{k_m^m}$

$$\frac{a_1}{10^{1!}} + \frac{a_2}{10^{2!}} + \frac{a_3}{10^{3!}} + \dots \qquad a_1, a_2, a_3 \dots = 1 \text{ or } 2$$

Fourier's dream (~ 1807)

"Any" 2π-periodic function $f: \mathbb{R} \to \mathbb{C}$ can be

written as
$$f(x) \cong \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$
$$\left(= \sum_{n=-\infty}^{\infty} c_n e^{inx} = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{inx} \right) \quad (*)$$

where
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad (n \geq 0)$$

(Euler)
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad (n \geq 1) \qquad (**)$$

Fourier coefficients of f.
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx. \qquad n \in \mathbb{Z}.$$

Dirichlet 1829
Riemann 1854 (published 1867)

Q.1 $\cong$ in what sense? (in $(*)$)

Heine 1870
Cantor (1870-72)

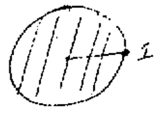
Q.2 Is $(*)$ unique?

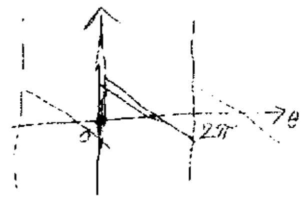
Q.3 In what sense are integrals taken in $(**)$.

If in $(*)$ uniform cgce then all is easy (probably Heine):
f has to be conts. & integrals are Cauchy integrals.
& coefficients are unique.

$$\log(1-z) = -\left(z + \tfrac{1}{2} z^2 + \cdots\right) \qquad z = re^{i\theta};\quad 0 \le r \le 1$$

$$|z| \le 1,\ z \ne 1 \qquad 0 < \theta < 2\pi \text{ if } r = 1$$





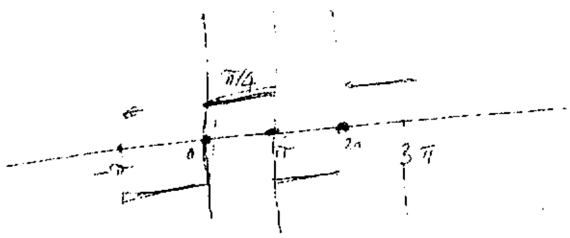
Now (taking real & imaginary parts)

$$(1)\quad \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \begin{cases} \frac{1}{2}(\pi - \theta) & \text{if } \theta \in \mathbb{R} \setminus 2\pi\mathbb{Z} \\ 0 & \text{if } \theta \in 2\pi\mathbb{Z} \end{cases}$$

$$(2)\quad \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = \log\left(\frac{1}{2\sin\frac{\theta}{2}}\right) \qquad \theta \ne \underset{k\in\mathbb{Z}}{2k\pi} \in 2\pi\mathbb{Z}$$

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \cdots = \begin{cases} \frac{\pi}{4}\,\mathrm{sgn}(\theta) & \text{if } -\pi < \theta < \pi \\ 0 & \text{if } \theta \in \pi\mathbb{Z} \\ & \left(\theta = \underset{k\in\mathbb{Z}}{k\pi}\right) \end{cases}$$

$$\mathrm{sgn}\,\theta = \begin{cases} 1 & \text{if } \theta > 0 \\ 0 & \text{if } \theta = 0 \\ -1 & \text{if } \theta < 0 \end{cases}$$



$$\text{By integration of (1)} \quad \sum_{n=1}^{\infty} \frac{1 - \cos n\theta}{n^2} = \frac{\pi\theta}{2} - \frac{1}{4}\theta^2 \qquad 0 \le \theta \le 2\pi$$

$$\text{From (2)} \quad \int_0^{\pi} \log\left(2\sin\tfrac{\theta}{2}\right) d\theta = 0 \qquad (3)$$

$$\text{whence } \int_0^{\pi/2} \theta\cot\theta\, d\theta \underset{(\text{parts})}{=} -\int_0^{\pi/2} \log\sin\theta\, d\theta = \frac{1}{2}\pi\log 2 \quad (\text{from } (3))$$

(See next page)

$$\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \qquad |z| \leq 1,\ z \neq -1$$

Take $z = e^{it}$, $-\pi < t < \pi$;

$$1 + e^{it} = (1 + \cos t) + i \sin t = 2 \cos \frac{t}{2} \left(\cos \frac{t}{2} + i \sin \frac{t}{2} \right)$$

So $\ln(1 + e^{it}) = \ln\left(2 \cos \frac{t}{2}\right) + i \frac{t}{2}$

and by taking real & imaginary parts

$$\begin{cases} \ln\left(2 \cos \frac{t}{2}\right) = \cos t - \frac{\cos 2t}{2} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nt}{n} \\ \frac{t}{2} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nt}{n} \end{cases} \qquad (-\pi < t < \pi)$$

Put $\theta = \pi - t$, $0 < \theta < 2\pi$, then we get $\begin{cases} \cos \frac{t}{2} = \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin \frac{\theta}{2} \\ \cos n(\pi - \theta) = (-1)^n \cos n\theta \\ \sin n(\pi - \theta) = (-1)^{n+1} \sin n\theta \end{cases}$

&

$$\begin{cases} \ln\left(\frac{1}{2 \sin \frac{\theta}{2}}\right) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} \\ \frac{\pi - \theta}{2} = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} \end{cases} \qquad 0 < \theta < 2\pi$$

etc.

So

$$\frac{\pi - \theta}{2} = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}, \quad 0 < \theta < \pi$$

$$\frac{\theta}{2} = \sum_{r=1}^{\infty} (-1)^{n-1} \frac{\sin n\theta}{n}, \quad 0 < \theta < \pi$$

hence

$$\frac{\pi}{2} = 2 \sum_{k=0}^{\infty} \frac{\sin(2k+1)\theta}{2k+1}, \quad 0 < \theta < \pi$$

$$\therefore \frac{\pi}{4} = \sin \theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots, \quad 0 < \theta < \pi$$

Cantor's uniqueness proof

Let $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x)$ TS $\forall x$

Put $F(x) = \frac{1}{4} a_0 x^2 - \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}$ (Riemann)

Then F is continuous (because $a_n, b_n \to 0$ (Cantor))

and

$$\frac{F(x+2h) + F(x-2h) - 2F(x)}{4h^2} \xrightarrow{h \to 0} f(x)$$

(for every x s.t. TS converges). Also $\frac{F(x+2h) + F(x-2h) - 2F(x)}{h} \xrightarrow{h\to 0} 0$

Schwarz's thm. If $\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \xrightarrow{h \to 0} 0$ then $F(x) = \alpha x + \beta$

If $f \equiv 0$ then $F(x) = \alpha x + \beta = \frac{1}{4} a_0 x^2 - \sum_n \frac{A_n(x)}{n^2}$

whence $\sum_{n=1}^{\infty} \frac{A_n(x)}{n^2} = \frac{1}{4} a_0 x^2 - \alpha x - \beta \Rightarrow a_0 = \alpha = 0$
then $\beta = 0$ by integration
using unif. cgce.

Same argument for finite no of exceptions.

True for denumerable exceptions

diagonal process $\quad \text{card } \mathbb{N} < \text{card } \mathbb{N}^{\mathbb{N}} \iff$ if $f_n : \mathbb{N} \to \mathbb{N}$, $n \geq 0$

$\exists\ f : \mathbb{N} \to \mathbb{N}$, $f \neq f_n\ \forall n$

Let $\quad f_0(0),\ f_0(1), \dots \quad = (f_0 : \mathbb{N} \to \mathbb{N})$

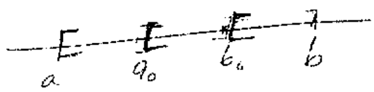
$\quad f_1(0),\ f_1(1), \dots \quad = (f_1 : \mathbb{N} \to \mathbb{N})$

$\quad \dots$

define $\quad f(n) = f_n(n) + 1, \quad n \geq 0$; then $f \neq f_n\ \forall n$

$\text{Card } \mathbb{N} < \text{card } \mathbb{R}$

if $x_0, x_1, x_2, \dots$ in $\mathbb{R}$ $\quad \exists\ x \neq x_n\ \forall n$, $x \in [a,b] \quad \forall [a,b]$



Proof. $\exists [a_0, b_0]$ s.t. $x_0 \notin [a_0, b_0] \subset [a, b]$

$\exists [a_1, b_1]$ s.t. $x_1 \notin [a_1, b_1] \subset [a_0, b_0]$

$b_0 - a_0 = \frac{1}{3}(b-a)$

$b_1 - a_1 = \frac{1}{9}(b-a)$

$\vdots$

$x_n \notin [a_n, b_n] \subset [a_{n-1}, b_{n-1}]$

$b_n - a_n = \frac{1}{3^{n+1}}(b-a)$

$\bigcap_n [a_n, b_n] = \{x\} \qquad x \neq x_n$ since $x_n \notin [a_n, b_n]$

$\text{Card } A < \text{card } \mathcal{P}(A)$

Proof: Suppose $\exists\ h : A \to \mathcal{P}(A)$ a bijection ; put $S = \{a \in A :\ a \notin h(a)\} \subset A$

$S \in \mathcal{P}(A)$

Then $\exists\ \alpha \in A$ s.t. $h(\alpha) = S$

If $\alpha \in S$ then $\alpha \notin h(\alpha) = S$ contradiction

If $\alpha \notin S$ then $\alpha \in h(\alpha) = S$ contradiction. So h cannot exist

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$\text{Card}(A \times A) = \text{card } A \quad \forall A$ infinite

Harward 1905

Hessenberg 1906

$\text{Card}(A \cup B) = \text{card } A$ if $\text{card } B \leq \text{card } A$

modes of cgce of FS

1) everywhere cgce or (conf. cgce everywhere) uniqueness (by Cantor)

2) cgce a.e. given any f meas. $\exists c_n$ s.t. $\sum c_n e^{inx} \to f(x)$ a.e. but not necessarily uniquely

3) cgce in L^2 (L^p) if $\sum |c_n|^2 < \infty$ then $\exists f \in L^2$ & vice-versa.

4) cgce in distribution.
if $\{c_n\}$ any sequence of polynomial growth then $\exists!$ dist f and vice-versa.

5) Suppose $f(x) = \sum_n c_n e^{inx}$ $\forall x$; then f is measurable (and of Baire class 1) is always Denjoy int. & F.S. of f is $\sum c_n e^{inx}$.
f need not by bdd. indeed Lebesgue integrable.