Lindeberg (1876 – 1932)
studied in Helsinki, Paris (1900 diff. eqns., calc. of vars.)
with Lindelöf Ernst.

1905 adjunkt in maths. in Helsinki, ~1920, 1922 famous thm.

later active in statistics, actuarial work

Math. Z. 15 (1922) 211 – 225 "Eine neue Herleitung des Exponentialgesetzes
in der ~~[illegible]~~ Wahrscheinlichkeitsrechnung."

de Moivre 1738
Laplace 1812
Chebyshev 1887
Markov 1898
Lyapunov 1900 - 1901
Lévy 1922
Feller 1936

Lévy (1886 – 1971)
A.A. Markov (1856 – 1922)
Lyapounov (1857 – 1918)
Chebyshev (1821 – 1894)
Laplace (1749 – 1827)
Cauchy (1789 – 1857)
de Moivre (1667 – 1754)
Poisson (1781 – 1840)
Bessel (1784 – 1846)

Finlande (~ 338'000 km², 5.2 million)
période suédoise — 1157 → 1550 (Vasa) → 1710-1721 (Pierre le grand)
(napoleonic wars) période russe 1809 (grand duché de l'Empire russe)
l'indépendance 1917

Lindeberg: son of a teacher at Poly, well-to-do family
1640 foundation of Academia Aboensis, Turku (Åbo)
1828 move to Helsinki (after fire in Turku)
Mittag-Leffler (1846 – 1927) Helsinki (1877 – 1881)

Theorem Let $\alpha_1, \dots, \alpha_n$ be n prob. meas. on $\mathbb{R}$ with

$M(\alpha_j) = 0$, $V(\alpha_j) = a_j^2$, $c(\alpha_j) = c_j < \infty$, $1 \le j \le n$; let

$$b_n^2 = a_1^2 + \cdots + a_n^2, \quad d_n^3 = c_1^3 + \cdots + c_n^3,$$

If $\mu = \alpha_1 * \cdots * \alpha_n$, $\nu = N(0; b_n^2)$ then

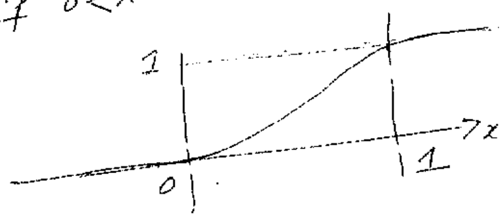
$$\sup_t |F_\mu(t) - F_\nu(t)| \le \kappa \left(\frac{d_n}{b_n}\right)^{3/4}$$

(Hausdorff Lecture Notes 1923, 1931)

(where $0 < \kappa < \infty$ is an universal constant

Proof Fix $h : \mathbb{R} \to \mathbb{R}$, $h \in \mathscr{C}_b^{(3)}(\mathbb{R})$ such that

$$h(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \geqslant 1 \end{cases}, \quad 0 < h(x) < 1 \text{ if } 0 < x < 1$$



$$m = \| h^{(3)} \| < \infty$$

(e.g. $h(x) = \int_0^x t^3(1-t)^3\,dt \Big/ \int_0^1 t^3(1-t)^3\,dt$, $0 < x < 1$.

Fix $0 < l < \infty$; put $g(x) = h(x/l)$; then $0 \le g \le 1$

and $g^{(3)}(x) = \frac{1}{l^3} h^{(3)}(x/l)$; $\| g^{(3)} \| = \frac{m}{l^3}$

From (7) with $\mu = \alpha_1 * \cdots * \alpha_n$,

$$\| T_\mu g - T_\nu g \| \le \frac{m}{2l^3} \sum_{j=1}^n c_j^3 = \frac{m}{2l^3} d_n^3$$

From (8) (with $\alpha = \mu$, $\beta = \nu$)

$$| F_\mu(t) - F_\nu(t) | \le \frac{m}{2l^3} d_n^3 + \frac{l}{b_n \sqrt{2\pi}} \qquad (*)$$

Now take $\ell = b_n^{1/4} d_n^{3/4}$ (essentially ~~[illegible]~~ minimizes rhs of $(*)$)

then $\frac{\ell}{b_n} = \left(\frac{d_n}{b_n}\right)^{3/4}$, $\frac{d_n}{\ell} = \left(\frac{d_n}{b_n}\right)^{3/4}$ &

$$|F_\mu(t) - F_\nu(t)| \le \frac{m}{2}\left(\frac{d_n}{b_n}\right)^{3/4} + \frac{1}{\sqrt{2\pi}}\left(\frac{d_n}{b_n}\right)^{3/4}$$

$$= \kappa \left(\frac{d_n}{b_n}\right)^{3/4}$$

(with $\kappa = \frac{m}{2} + \frac{1}{\sqrt{2\pi}}$ some well determined pos. const.

(N.B. $m < 54$ is easy; so $\kappa < 28$).

<u>Remark</u> The exact value of κ is not interesting since the order of cgce in the theorem is very far from being optimal.

(Thus ~~[illegible]~~ in the iid case $a_j^2 = a^2$, $c_j^3 = c^3$, $d_n^3 = n c^3$, $b_n^2 = n a^2$

and $\frac{d_n}{b_n} = \frac{c^3}{a^2} \frac{n^{1/3}}{n^{1/2}} = \frac{c^3}{a^2} n^{-1/6}$

~~an~~ $\left(\frac{d_n}{b_n}\right)^{3/4} = \text{const} \cdot n^{-\frac{1}{6}\frac{3}{4}} = \text{const} \cdot n^{-1/8}$ (far from $n^{-1/2}$ the optimal rate).

$T_\alpha : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{B}(\mathbb{R})$, $T_\alpha g(t) = \int_{-\infty}^{\infty} g(t-x)\, d\alpha(x)$

$\|g\| = \sup_{x \in \mathbb{R}} |g(x)|$

α = prob. meas. on $\mathbb{R}$

$M(\alpha) = \int_{-\infty}^{\infty} (x)\, d\alpha(x)$

$V(\alpha) = \operatorname{var} \alpha$

$c^3(\alpha) = \int_{-\infty}^{\infty} |x|^3 d\alpha(x)$

$F_\alpha(t) = \alpha(]-\infty, t])$

(1) T_α is linear & $\|T_\alpha\| \leq 1$

(2) $T_\alpha T_\beta = T_\beta T_\alpha = T_{\alpha * \beta}$

(3) $\|A_1 \cdots A_n - B_1 \cdots B_n\| \leq \sum_{j=1}^{n} \|A_j - B_j\|$

if $\{A_1, \cdots A_n, B_1, \cdots, B_n\}$ is a commuting family of contractions.

(4) If $g \in \mathcal{C}^{(k)}(\mathbb{R})$ then $T_\alpha g \in \mathcal{C}^{(k)}(\mathbb{R})$ & $\|(T_\alpha g)^{(j)}\| \leq \|g^{(j)}\|$ $j = 0, 1, \cdots, k$.

(5) If $M(\alpha) = M(\beta) = 0$, $V(\alpha) = V(\beta)$ & $C(\alpha), C(\beta) < \infty$ then

$$\|T_\alpha g - T_\beta g\| \leq \frac{1}{6} \|g^{(3)}\| \left(C^3(\alpha) + c^3(\beta) \right), \quad g \in \mathcal{C}^{(3)}(\mathbb{R})$$

(6) If $M(\alpha_j) = M(\beta_j) = 0$, $V(\alpha_j) = V(\beta_j)$, $c^3(\alpha_j) < \infty$, $c^3(\beta_j) < \infty$ $1 \leq j \leq n$

then $\forall g \in \mathcal{C}^{(3)}(\mathbb{R})$

$$\| T_{\underbrace{\alpha_1 * \cdots * \alpha_n}_{\mu}} g - T_{\underbrace{\beta_1 * \cdots * \beta_n}_{\nu}} g \| \leq \frac{1}{6} \|g^{(3)}\| \sum_{j=1}^{n} \{ c^3(\alpha_j) + c^3(\beta_j) \}$$

(7) If $\beta_j = N(0; a_j^2)$, $a_j^2 = V(\alpha_j)$ then $c^3(\beta_j) < 2\, c^3(\alpha_j)$

and (6) becomes

$$\| T_\mu g - T_\nu g \| \leq \frac{1}{2} \|g^{(3)}\| \sum_{j=1}^{n} c^3(\alpha_j)$$

(8) Let $g \in B(\mathbb{R})$, $0 \le g \le 1$, $g(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge \ell \end{cases}$; then

for any two prob. meas. α, β

(i) $T_\alpha g(t) \le F_\alpha(t) \le T_\alpha g(t+\ell) \quad \forall t \in \mathbb{R}$

$T_\beta g(t) \le F_\beta(t) \le T_\beta g(t+\ell)$

(ii) $F_\alpha(t) - F_\beta(t) \le \| T_\alpha g - T_\beta g \| + F_\beta(t+\ell) - F_\beta(t)$

$F_\beta(t) - F_\alpha(t) \le \| T_\alpha g - T_\beta g \| + F_\beta(t) - F_\beta(t-\ell)$

(iii) If $\beta = N(0, b^2)$ then

$$| F_\alpha(t) - F_\beta(t) | \le \| T_\alpha g - T_\beta g \| + \frac{\ell}{b\sqrt{2\pi}}$$

Proof of (2) $\quad T_\alpha T_\beta = T_{\alpha*\beta} = T_{\beta*\alpha}$

$$T_{\alpha*\beta}\, g(t) = \int_{-\infty}^{\infty} g(t-x)\, d(\alpha*\beta)(x)$$

For any bdd. Borel $h : \mathbb{R} \to \mathbb{C}$

$$(*) \quad \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(x+y)\, d\alpha(y) \right) d\beta(y) = \int_{-\infty}^{\infty} h(z)\, d(\alpha*\beta)(z)$$
$$= \int_{-\infty}^{\infty} h(z)\, d(\beta*\alpha)(z)$$

(Fubini)

Since $(*)$ true for $h = 1_A$, $A \subset \mathbb{R}$ for here $(*)$ reduces to

$$\int_{-\infty}^{\infty} \alpha(A-x)\, d\beta(x) = (\alpha*\beta)(A) \quad \text{def. of } \alpha*\beta$$

So

$$T_{\alpha*\beta}\, g(t) = \int_{-\infty}^{\infty} \underbrace{g(t-x)}_{h(x)}\, d(\alpha*\beta)(x)$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(t-u-v)\, d\alpha(v) \right) d\beta(u)$$

$$= \int_{-\infty}^{\infty} T_\alpha g(t-u)\, d\beta(u)$$

$$= T_\beta \left(T_\alpha g \right)(t)$$

Proof of (3)

Let $P_n = A_1 \cdots A_n$, $Q_n = B_1 B_2 \cdots B_n$; then

$$P_{n+1} - Q_{n+1} = P_n A_{n+1} - Q_n B_{n+1}$$

$$= A_{n+1} P_n - B_{n+1} P_n + B_{n+1} P_n - B_{n+1} Q_n$$

and

$$\| P_{n+1} - Q_{n+1} \| \leq \| (A_{n+1} - B_{n+1}) P_n \| + \| B_{n+1} (P_n - Q_n) \|$$

$$\leq \| A_{n+1} - B_{n+1} \| + \| P_n - Q_n \|$$

whence we conclude by induction.

Proof of (5)

$M(\alpha) = 0$, $M(\beta) = 0$, $V(\alpha) = V(\beta) = \sigma^2$

$$g(t-x) = g(t) - x\, g'(t) + \frac{x^2}{2!} g''(t) - \frac{x^3}{6} R(t,x) \qquad |R(t,x)| \leq \|g^{(3)}\|$$

$$T_\alpha g(t) = \int_{-\infty}^{\infty} g(t-x)\, d\alpha(x) = g(t) + \frac{g''(t)}{2} \sigma^2 - \frac{1}{6} \int_{-\infty}^{\infty} x^3 R(t,x)\, d\alpha(x)$$

$$\left| T_\alpha g(t) - g(t) - \frac{g''(t)}{2} \sigma^2 \right| \leq \frac{1}{6} \| g^{(3)} \| \int_{-\infty}^{\infty} |x|^3\, d\alpha(x) = \frac{1}{6} \| g^{(3)} \|\, C^3(\alpha)$$

Similarly

$$\left| T_\beta g(t) - g(t) - \frac{g''(t)}{2} \sigma^2 \right| \leq \frac{1}{6} \| g^{(3)} \|\, C^3(\beta)$$

whence

$$\left| T_\alpha g(t) - T_\beta g(t) \right| \leq \frac{1}{6} \| g^{(3)} \| \left(C^3(\alpha) + C^3(\beta) \right)$$

Proof of (7)

If $\quad \beta = N(0, \sigma^2), \begin{cases} M(\alpha) = 0 \\ V(\alpha) = \sigma^2 \\ C(\alpha) < \infty \end{cases} \quad$ then

$$c^3(\beta) < 2\, c^3(\alpha)$$

since

$$c^3(\beta) = \int_{-\infty}^{\infty} |x|^3 \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}}\, dx$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}}\, \sigma^3$$

& $$\sigma^3 = \left(\int_{-\infty}^{\infty} |x|^2\, d\alpha(x)\right)^{3/2} \underset{\text{Jensen}}{\leq} \int_{-\infty}^{\infty} |x|^3\, d\alpha(x) = c^3(\alpha)$$

So $$c^3(\beta) = \frac{2\sqrt{2}}{\sqrt{\pi}}\, \sigma^3 \leq \frac{2\sqrt{2}}{\sqrt{\pi}}\, c^3(\alpha) < 2\, c^3(\alpha)$$

Theorem: Let $\alpha_1, \ldots, \alpha_n$ be n prob. meas. in $\mathbb{R}$

with $M(\alpha_j)=0$, $V(\alpha_j)=a_j^2$, $a_1^2+\cdots+a_n^2=1$ $(1\leq j\leq n)$

Let $$S(x)=\begin{cases}|x|^3 & \text{if } |x|<1\\ x^2 & \text{if } |x|\geq 1\end{cases}\quad ; L=L(n)=\sum_{j=1}^{n}\int_{-\infty}^{\infty}S(x)\,d\alpha_j(x)$$

If $\mu=\alpha_1 * \cdots * \alpha_n$, $\nu=N(0,1)$ then

$$\sup_t |F_\mu(t)-F_\nu(t)| \leq \varkappa L^{1/4}$$ (Lindeberg 1922)

where $0<\varkappa<\infty$ is an universal constant.

N.B. In terms of real r.v.'s $x_1, \ldots, x_n$ indep., ~~$E(x_j)=a_j$~~ $V(x_j)=a_j^2$, $M(x_j)=0$

$b_n^2 = a_1^2+\cdots+a_n^2$

we have

$$\sup_t \left| \mathbb{P}\left(\frac{x_1+\cdots+x_n}{b_n}\leq t\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-u^2/2}\,du \right| \leq \varkappa L(n)^{1/4}$$

where $$L(n)=\sum_{j=1}^{n}\mathbb{E}\, S\left(\frac{x_j}{b_n}\right)$$

The Lindeberg condition is: $L(n)\to 0$ as $n\to\infty$, under this condition we have

$$\sup_t \left| \mathbb{P}\left(\frac{x_1+\cdots+x_n}{b_n}\leq t\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-u^2/2}\,du \right| \xrightarrow[\text{as } n\to\infty]{} 0.$$

which is the Lindeberg CLT.

Let $h \in \mathcal{C}_b^{(3)}(\mathbb{R})$; $h(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1 \end{cases}$, $0 \le h(x) \le 1$ and

$\|h^{(3)}\| = m$; then $\|h^{(2)}\| \le m$ since $h^{(2)}(x) = \int_0^x h^{(3)}(t)\,dt$

$|h^{(2)}(x)| \le mx$, $0 < x < 1$

and $h^{(2)}(x) = 0$ if $x \le 0$ & $x \ge$

$M(\alpha) = 0$
$V(\alpha) = a^2$

Write $h(t-x) = h(t) - x\,h'(t) + \frac{x^2}{2} h''(t) + R(t,x)$

Then

$$T_\alpha h(t) = \int_{-\infty}^{\infty} h(t-x)\,d\alpha(x) = h(t) + \frac{a^2}{2} h''(t) + \int_{-\infty}^{\infty} R(t,x)\,d\alpha(x)$$

Now

$$\left| \int_{-\infty}^{\infty} R(t,x)\,d\alpha(x) \right| \le \int_{|x|<1} + \int_{|x|\ge 1} |R(t,x)|\,d\alpha(x)$$

$$\le m\int_{|x|<1} \frac{|x|^3}{6}\,d\alpha(x) + m\int_{|x|\ge 1} |x|^2\,d\alpha(x) \le m\int_{-\infty}^{\infty} S(x)\,d\alpha(x)$$

since we may write

$$R(t,x) = \begin{cases} \frac{x^3}{6} h^{(3)}(t-\theta) \quad (\theta \text{ between } 0, x) & \text{if } |x| < 1 \\ \frac{x^2}{2}\left(h^{(2)}(t-\theta) - h^{(2)}(t)\right) & \text{if } |x| \ge 1 \end{cases}$$

so

$$|R(t,x)| \le \begin{cases} |x|^3 \cdot \frac{m}{6} & \text{if } |x| < 1 \\ x^2\, m & \text{if } |x| \ge 1 \end{cases}$$

Thus

$$\left| T_\alpha h(t) - h(t) - \frac{a^2}{2} h''(t) \right| \le m \int_{-\infty}^{\infty} S(x)\,d\alpha(x)$$

Similarly, if $\beta = N(0, a^2)$,

$$\left| T_\beta h(t) - h(t) - \frac{a^2}{2} h''(t) \right| \le m \int_{-\infty}^{\infty} S(x)\,d\beta(x) \text{ or } \frac{1}{3}\sqrt{\frac{2}{\pi}}\, m a^3$$

by using 3rd moment. See next page

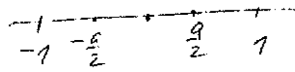
and

$$\left| T_\alpha h(t) - T_\beta h(t) \right| \le m\left(\int_{-\infty}^{\infty} S\,d\alpha + \sqrt{\frac{2}{\pi}} \cdot \frac{1}{3} a^3 \right)$$

Now if $a < 1$, $a^3 < a^2$ and $\int_{-\infty}^{\infty} S\,d\alpha > \frac{a^3}{4}$

since either $\int_{|x|\geq 1} x^2\,d\alpha(x) \geq \frac{a^2}{4}$ in which case $\int_{-\infty}^{\infty} S\,d\alpha \geq \int_{|x|\geq 1} x^2\,d\alpha(x) \geq \frac{a^2}{4} > \frac{a^3}{4}$

or $\int_{|x|\geq 1} x^2\,d\alpha(x) < \frac{a^2}{4}$ in which case $\int_{|x|<1} x^2\,d\alpha(x) \geq \frac{3a^2}{4}$



& $$\int_{\frac{a}{2}<|x|<1} x^2\,d\alpha(x) \geq \frac{3a^2}{4} - \int_{|x|\leq \frac{a}{2}} x^2\,d\alpha(x)$$

$$\geq \frac{3a^2}{4} - \frac{a^2}{4} = \frac{a^2}{2}$$

whence $$\int_{|x|<1} |x|^3\,d\alpha(x) \geq \int_{\frac{a}{2}<|x|<1} |x|^3\,d\alpha(x) \geq \frac{a}{2}\int_{\frac{a}{2}<|x|<1} x^2\,d\alpha(x) \geq \frac{a^3}{4}$$

& $$\int_{-\infty}^{\infty} S(x)\,d\alpha(x) \geq \int_{|x|<1} |x|^3\,d\alpha(x) \geq \frac{a^3}{4}$$

i.e. $a^3 \leq 4\int_{-\infty}^{\infty} S\,d\alpha$

& $$|T_\alpha h(t) - T_\beta h(t)| \leq m\int_{-\infty}^{\infty} S\,d\alpha \left(1 + \frac{4}{3}\sqrt{\frac{2}{\pi}}\right) \leq 3m\int_{-\infty}^{\infty} S\,d\alpha$$

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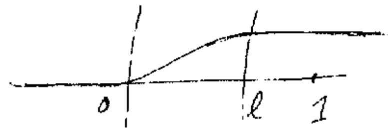
* $$\left| T_\beta h(t) - h(t) - \frac{a^2}{2}h''(t)\right| \leq \left|\int_{-\infty}^{\infty} \frac{x^3}{6} h^{(3)}(t-\theta)\,d\beta(x)\right|$$

$$\leq \frac{m}{6}\int_{-\infty}^{\infty} |x|^3\,d\beta(x) = \frac{m}{6}\cdot\frac{2\sqrt{2}}{\sqrt{\pi}}a^3 < \frac{m}{3}a^3\sqrt{\frac{2}{\pi}}$$

Fix $0<l\le 1$; $g(t)=h\left(\frac{t}{l}\right)$; then



$g(t)=0$ if $t\le 0$, $g(t)=1$ if $t\ge l$

$0\le g\le 1$, $g^{(3)}(t)=\frac{1}{l^3}h^{(3)}\left(\frac{t}{l}\right)$, $\|g^{(3)}\|=\frac{m}{l^3}$

and we get

$$|T_\alpha g(t)-T_\beta g(t)|\le \frac{3m}{l^3}\int_{-\infty}^{\infty} s\,d\alpha$$

Apply this to $\alpha=\mu=\alpha_1*\dots*\alpha_n$; $M(\alpha_j)=0$, $V(\alpha_j)=a_j^2$, $a_1^2+\dots+a_n^2=1$

$\beta=\nu=N(0,1)=\beta_1*\dots*\beta_n$, $\beta_j=N(0,a_j^2)$

Then

$$\|T_\mu g-T_\beta g\|\le \sum_{j=1}^{n}\|T_{\alpha_j}g-T_{\beta_j}g\|$$

$$\le \frac{3m}{l^3}\sum_{j=1}^{n}\int_{-\infty}^{\infty} s\,d\alpha_j$$

As before

$$|F_\mu(t)-F_\nu(t)|\le \frac{3m}{l^3}\sum_{j=1}^{n}\int_{-\infty}^{\infty} s\,d\alpha_j+\frac{1}{\sqrt{2\pi}}\,l$$

Put $l=\left(\sum_{j=1}^{n}\int_{-\infty}^{\infty} s\,d\alpha_j\right)^{1/4}\le 1$; this gives (put $L=\sum_{j=1}^{n}\int_{-\infty}^{\infty} s\,d\alpha_j$)

(since $\int_{-\infty}^{\infty} s\,d\alpha_j=\int_{|x|<1}|x|^3\,d\alpha_j+\int_{|x|\ge 1}|x|^2\,d\alpha_j\le\int_{|x|<1}|x|^2\,d\alpha_j+\int_{|x|\ge 1}|x|^2\,d\alpha_j=a_j^2$, $\sum_{j=1}^{n}a_j^2=1$)

$$|F_\mu(t)-F_\nu(t)|\le L^{1/4}\left(3m+\frac{1}{\sqrt{2\pi}}\right)=\varkappa L^{1/4}$$

Put $L(n) = \sum_{j=1}^{n} \mathbb{E}\, s\left(\frac{x_j}{b_n}\right)$; $L_\varepsilon(n) = \frac{1}{b_n^2} \sum_{j=1}^{n} \mathbb{E}\left(|x_j|^2 ; |x_j| \geq \varepsilon b_n\right)$

Then $L(n) \to 0$ as $n \to \infty$ $\iff$ $L_\varepsilon(n) \to 0$ as $n \to \infty$. $\forall \varepsilon > 0$

Proof. It suffices to consider $0 < \varepsilon < 1$ (since $\varepsilon < \varepsilon' \Rightarrow L_\varepsilon(n) \geq L_{\varepsilon'}(n)$)

If $0 < \varepsilon < 1$ & $x \geq \varepsilon$ then $\varepsilon x^2 \leq s(x) \leq x^2$

Note that

$$\frac{|x_j|}{b_n} \geq \varepsilon \Rightarrow \varepsilon \frac{x_j^2}{b_n^2} \leq s\left(\frac{x_j}{b_n}\right) \leq \frac{x_j^2}{b_n^2}$$

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since either $\frac{|x_j|}{b_n} \geq 1$ then $s\left(\frac{x_j}{b_n}\right) = \frac{x_j^2}{b_n^2} > \varepsilon \frac{x_j^2}{b_n^2}$

or $\varepsilon \leq \frac{|x_j|}{b_n} < 1$ then $s\left(\frac{x_j}{b_n}\right) = \left|\frac{x_j}{b_n}\right|^3 = \frac{x_j^2}{b_n^2} \cdot \frac{|x_j|}{b_n}$

Hence

$$\sum_{j=1}^{n} \mathbb{E}\left\{ s\left(\frac{x_j}{b_n}\right) ; |x_j| \geq \varepsilon b_n \right\} \geq \frac{\varepsilon}{b_n^2} \sum_{j=1}^{n} \mathbb{E}\left(|x_j|^2 ; |x_j| \geq \varepsilon b_n\right) = \varepsilon L_\varepsilon(n)$$

whence

$$L_\varepsilon(n) \leq \frac{1}{\varepsilon} L(n)$$

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Also

$$L(n) = \sum_{j=1}^{n} \mathbb{E}\left\{ s\left(\frac{x_j}{b_n}\right), |x_j| \geq \varepsilon b_n \right\} + \sum_{j=1}^{n} \mathbb{E}\left\{ s\left(\frac{x_j}{b_n}\right); |x_j| < \varepsilon b_n \right\}$$

$$\leq \sum_{j=1}^{n} \mathbb{E}\left\{ \frac{|x_j|^2}{b_n^2} ; |x_j| \geq \varepsilon b_n \right\} + \sum_{j=1}^{n} \mathbb{E}\left\{ \left|\frac{x_j}{b_n}\right|^3 ; |x_j| < \varepsilon b_n \right\}$$

$$\leq L_\varepsilon(n) + \underbrace{\frac{\varepsilon}{b_n^2} \sum_{j=1}^{n} \mathbb{E}\left(|x_j|^2 ; |x_j| < \varepsilon b_n\right)}_{\leq 1}$$

$$\frac{|x_j|^3}{b_n^3} = \frac{x_j^2 |x_j|}{b_n^3} < \frac{x_j^2 \varepsilon}{b_n^2}$$

$$\leq L_\varepsilon(n) + \varepsilon$$

Let $\quad d_n^3 = \sum_{j=1}^{n} \mathbb{E}|x_j|^3 \qquad b_n^2 = \sum_{j=1}^{n} \mathbb{E}\, x_j^2 \quad ;$

$$\mathbb{E}|x_j|^3 \geq \mathbb{E}\left(\underbrace{|x_j|^3}_{|x_j|\cdot x_j^2} \; ; \; |x_j| \geq \varepsilon b_n\right) \geq \varepsilon b_n \, \mathbb{E}\left(x_j^2 \, ; \, |x_j| \geq \varepsilon b_n\right)$$

So

$$d_n^3 \geq \varepsilon b_n \cdot b_n^2 \cdot \frac{1}{b_n^2} \sum_{j=1}^{n} \mathbb{E}\left(x_j^2 \, ; \, |x_j| \geq \varepsilon b_n\right) = \varepsilon b_n^3 L_\varepsilon(n)$$

i.e. $\qquad L_\varepsilon(n) \leq \frac{1}{\varepsilon} \left(\frac{d_n}{b_n}\right)^3$

Hence $\qquad \frac{d_n}{b_n} \xrightarrow{n\to\infty} 0 \quad \Rightarrow \quad L_\varepsilon(n) \xrightarrow{n\to\infty} 0 \qquad \forall \varepsilon > 0$.

Similarly, $(\delta > 0)$

$$\frac{\sum_{j=1}^{n} \mathbb{E}|x_j|^{2+\delta}}{b_n^{2+\delta}} \xrightarrow{n\to\infty} 0 \quad \Rightarrow \quad L_\varepsilon(n) \xrightarrow{n\to\infty} 0 \qquad \forall \varepsilon > 0$$

Since $\quad \mathbb{E}|x_j|^{2+\delta} \geq \mathbb{E}\left(|x_j|^{2+\delta} \, ; \, |x_j| \geq \varepsilon b_n\right) \geq (\varepsilon b_n)^\delta \, \mathbb{E}\left(|x_j|^2 , \, |x_j| \geq \varepsilon b_n\right)$

$$\sum_{j=1}^{n} \mathbb{E}|x_j|^{2+\delta} \geq (\varepsilon b_n)^\delta b_n^2 \cdot L_\varepsilon(n) \quad \text{i.e.} \quad L_\varepsilon(n) \leq \frac{\sum_{j=1}^{n} \mathbb{E}|x_j|^{2+\delta}}{\varepsilon^\delta b_n^{2+\delta}}$$

Special cases:

(1) $\{x_j\}$ i.i.d $\quad \mathbb{E} x_j^2 = a^2 < \infty$; then $b_n^2 = n a^2$ and then CLT

$\mathcal{L}(x) = \mathcal{L}(x_j)$

$$L_\varepsilon(n) = \frac{1}{n a^2} \sum_{j=1}^{n} \mathbb{E}\left(x^2 ; \ |x| \geq \varepsilon a \sqrt{n} \right)$$

Since $\mathbb{E} x^2 < \infty$, $\quad \lim_{n\to\infty} \mathbb{E}\left(x^2; \ |x| \geq \varepsilon a\sqrt{n}\right) = 0$

which implies that $\quad L_\varepsilon(n) \to 0$ as $n \to \infty \quad \forall \varepsilon > 0$.

(2) Suppose $|x_j| \leq M$, $\quad \sum_{j=1}^{n} a_j^2 = b_n^2 \to \infty \ (n\to\infty)$. Then CLT

In this case, $\forall \varepsilon > 0 \quad \mathbb{E}\left(|x_j|^2 ; \ |x_j| \geq \varepsilon b_n\right) = 0 \quad n \geq n_\varepsilon$

($\varepsilon b_n > M$ if $n \geq n_\varepsilon$)

So that $\displaystyle L_\varepsilon(n) = \frac{1}{b_n^2} \sum_{j=1}^{n_\varepsilon} \mathbb{E}\left(|x_j|^2; \ |x_j| \geq \varepsilon b_n\right) \xrightarrow[n\to\infty]{} 0$

More simply

$$\mathbb{E}|x_j|^3 \leq M \, \mathbb{E}|x_j|^2 = M a_j^2$$

$$\sum_{j=1}^{n} \mathbb{E}|x_j|^3 \leq M \cdot b_n^2, \qquad \frac{\sum_{j=1}^{n} \mathbb{E}|x_j|^3}{b_n^3} \leq \frac{M}{b_n} \xrightarrow{n\to\infty} 0$$

(3) Suppose $|x_j| \leq m_j$, $M_n = \max(m_1, m_2, \dots, m_n)$, $\mathbb{E} x_j^2 = a_j^2$

$$b_n^2 = \sum_{j=1}^{n} a_j^2$$

If $\displaystyle \frac{M_n}{b_n} \xrightarrow{n\to\infty} 0$ then CLT

Since $\displaystyle \sum_{j=1}^{n} \mathbb{E}|x_j|^3 \leq \sum_{j=1}^{n} m_j \cdot a_j^2 \leq M_n b_n^2, \quad \frac{\sum_{j=1}^{n} \mathbb{E}|x_j|^3}{b_n^3} \leq \frac{M_n}{b_n} \to 0$