200 Years of Least Squares Method

A. Abdulle and G. Wanner, Genève, Basel

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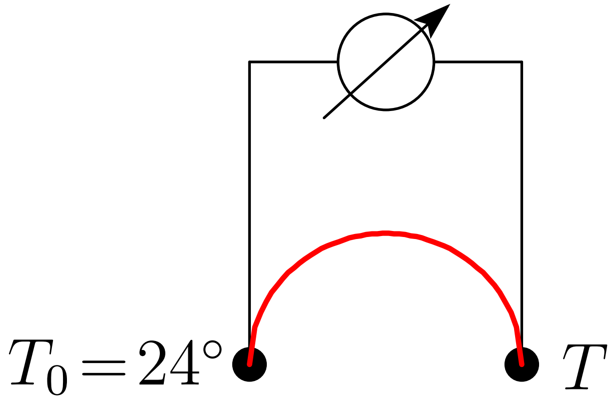
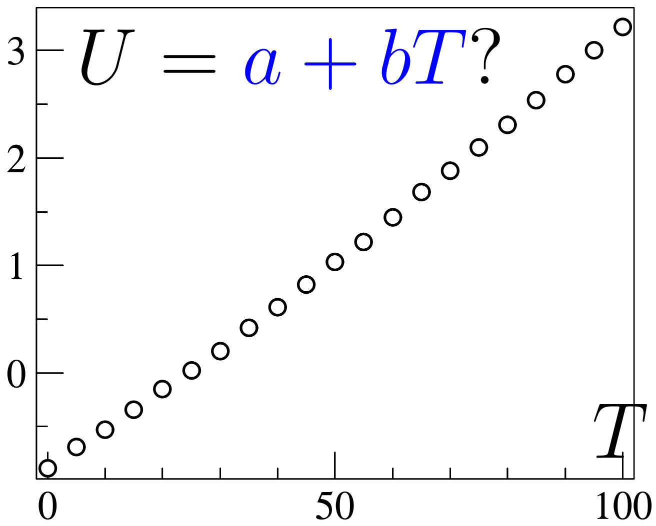
The method of least squares is the automobile of modern statistical analysis;. . .

(The first sentence of Stigler, *Gauss and the invention of least squares*, The Annals of Stat., 9, 1981)

Example: Thermojunction

T_i°C	U_i	T_i°C	U_i	T_i°C	U_i
0	−0.89	35	0.42	70	1.88
5	−0.69	40	0.61	75	2.10
10	−0.53	45	0.82	80	2.31
15	−0.34	50	1.03	85	2.54
20	−0.15	55	1.22	90	2.78
25	0.02	60	1.45	95	3.00
30	0.20	65	1.68	100	3.22





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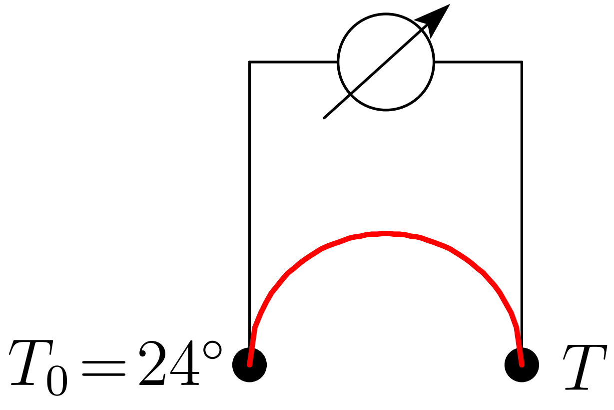
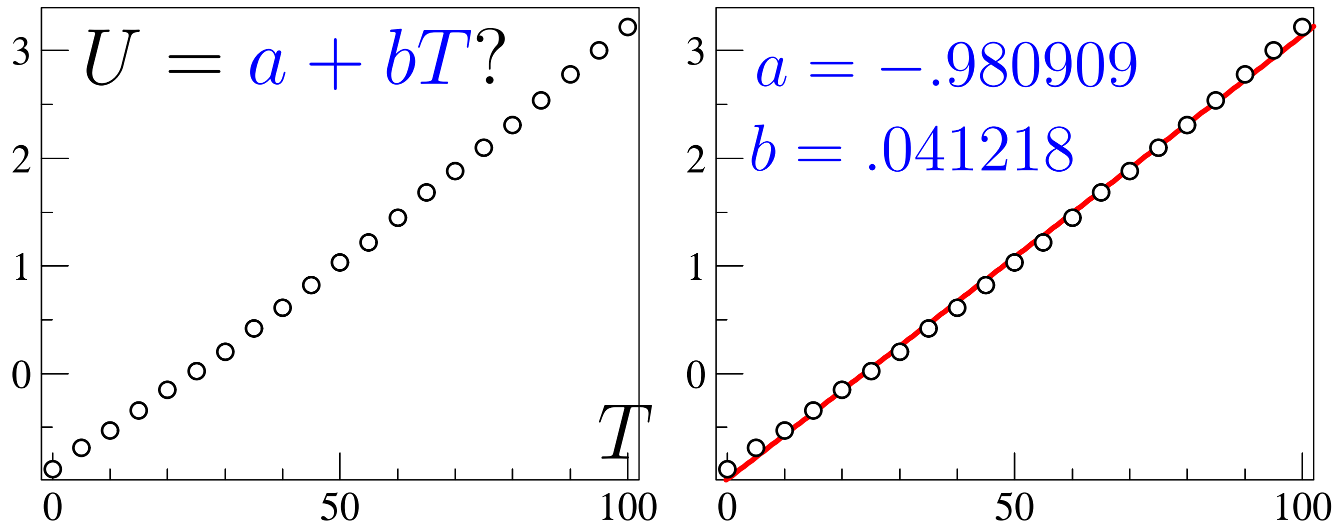


$$\sum_i (a + bT_i - U_i)^2 = \text{min!}$$



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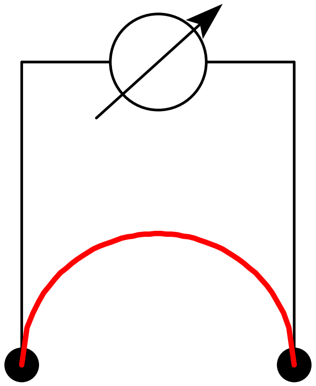
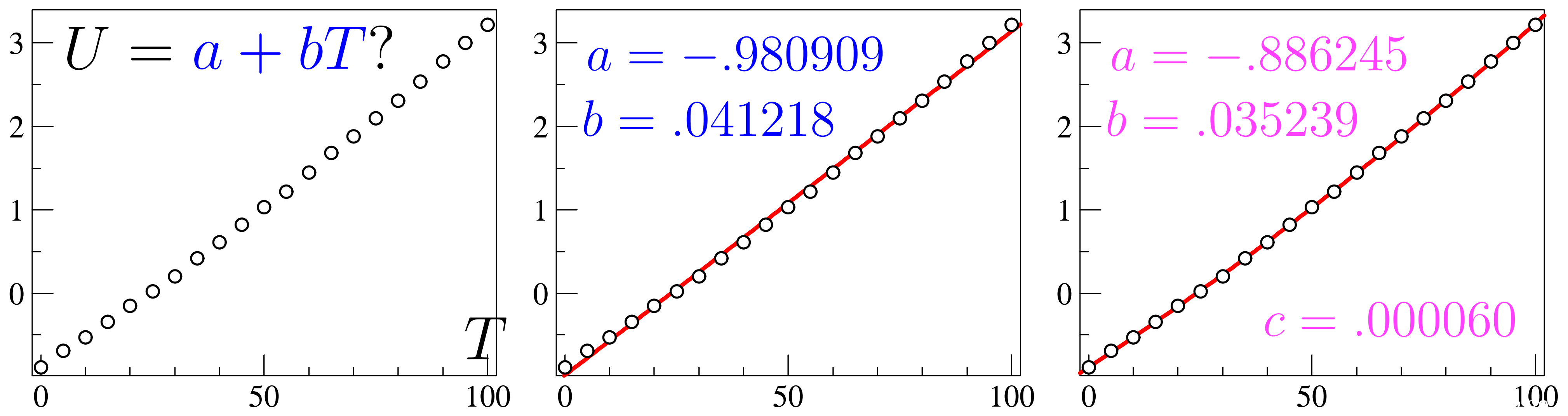
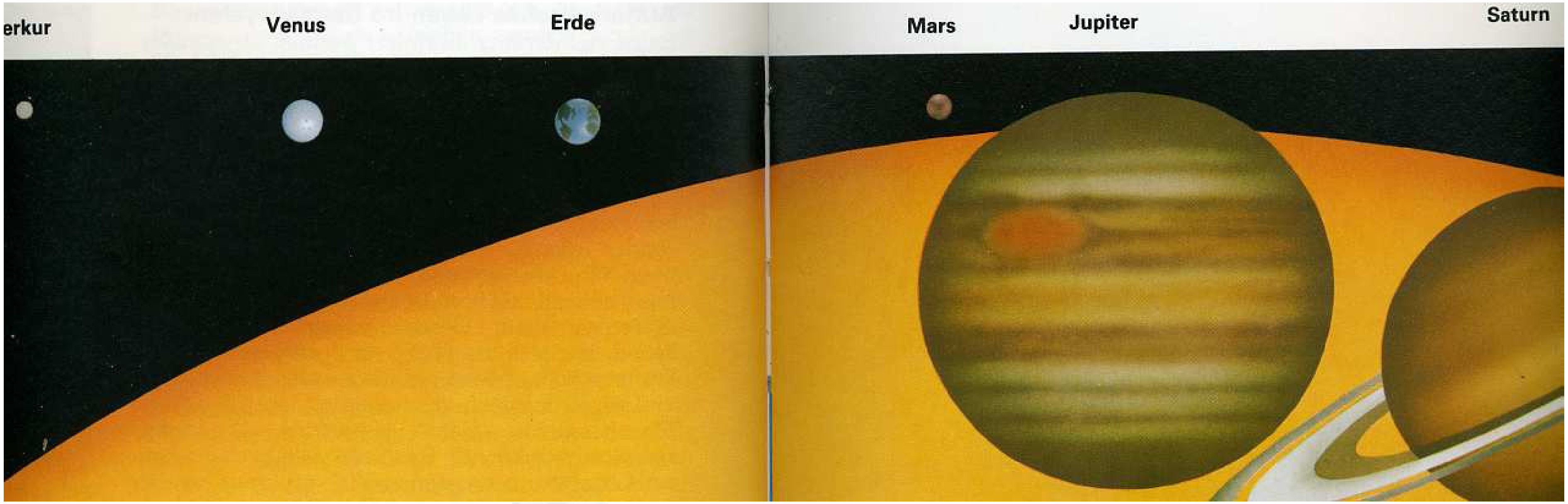


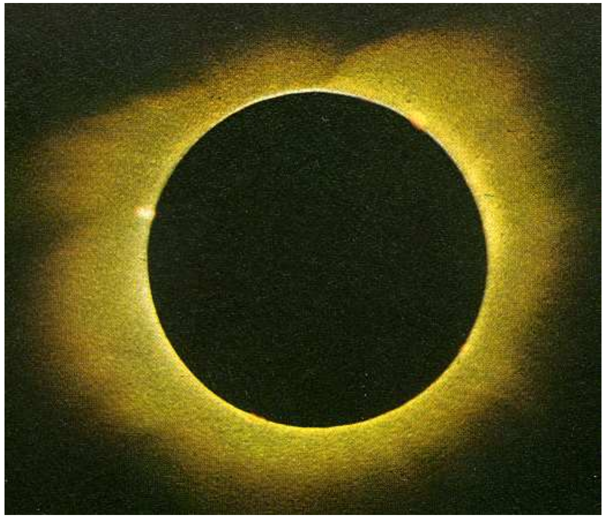
$$\sum_i (a + bT_i - U_i)^2 = \text{min!}$$

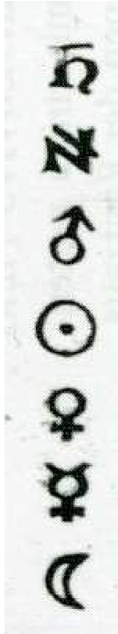
$$\sum_i (a + bT_i + cT_i^2 - U_i)^2 = \text{mi}$$



History.







Babylon. and Egypt. civilisation:

The Seven Heavenly Gods

Sunday, lunedi, martedi, mercoledi, giovedi, venerdi, Saturday.

... and then nothing new happened for more than 2000 years ...

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Sensation:

Sir William Herschel:
a German organist and amateur astronomer
living in England

discovered the 13th of march 1781 a new planet, through a huge telescope of his own construction. Herschel wanted to name ‘his’ new planet *Georgium sidus* (George’s star), in devotion to the British King, but Bode’s proposition *Uranus* (in Greek mythology the father of Saturnus), was felt less patriotic and became generally accepted.

The Rule of Titius-Bode.

> Das Daseyn dieses Planeten scheint insbesondere aus einem merkwürdigen Verhältniss zu folgen . . . Sollte der Urheber der Welt diesen Raum leer gelassen haben?
>
> (J.E. Bode, *Anleitung zur Kenntniss des gestirnten Himmels*, 6. Aufl., Berlin 1792, quoted in *Hegels Werke* 5, Anmerkungen p. 810)

$$0.4, \quad 0.4 + 0.3 = 0.7, \quad 0.4 + 2 \cdot 0.3 = 1 \text{ (the earth)}, \quad \ldots$$

$$0.4 + 2^{n-2} \cdot 0.3, \ldots .$$

For $n = 1, 2, 3, 4, 6, 7, 8$ approximate well the distances of the known planets.

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For $n = 1, 2, 3, 4, 6, 7, 8$ approximate well the distances of the known planets.

$n = 5$ is missing !! Is there a gap?? The creator has certainly put something there!!

The Thesis of Hegel.

Dissertatio philosophica de orbitis planetarum, Ienae MDCCCI.

Plato’s *Timæus*: the “Soul of the World” :

Moon	1
Sun	2
Venus	3
Mercury	4
Mars	8
Jupiter	9
Saturn	27

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"16 enim pro 8 quem legimus ponere liceat"

$\sqrt[3]{x^4}$ and ponamus $\sqrt[3]{3}$ for 1

$1.4 \quad 2.56 \quad 4.37 \quad 6.34 \quad 18.75 \quad 40.34 \quad 81$

"inter quartum et quintum locum magnum esse spatium"

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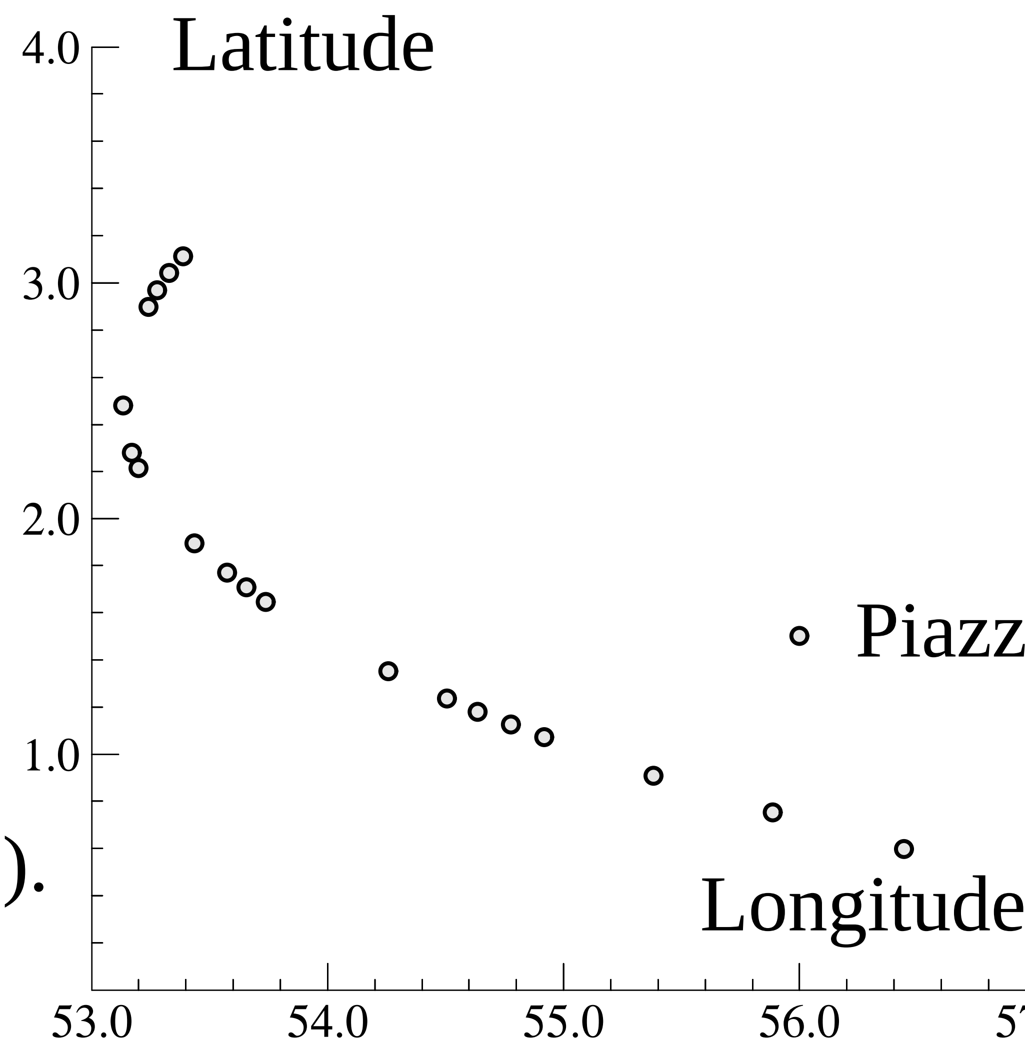
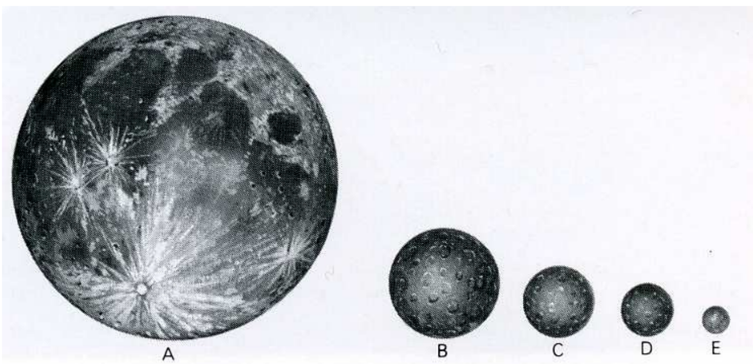
Sehen Sie sich doch nur bei den heutigen Philosophen um, bei Schelling, Hegel, Nees von Esenbeck und Consorten, stehen Ihnen nicht die Haare bei ihren Definitionen zu Berge?

(Brief von Gauss an Schumacher, 1. 11. 1844)

The Discovery of Piazzi.

On January 1, 1801, the Italian astronomer Giuseppe Piazzi discovered in the Taurus constell. a tiny little spot, followed its orbit until the 11th of February, when illness, bad weather, and the approaching Sun interrupted the observations. He named it *Ceres Ferdinandea* (Ferdinand is another King's name).





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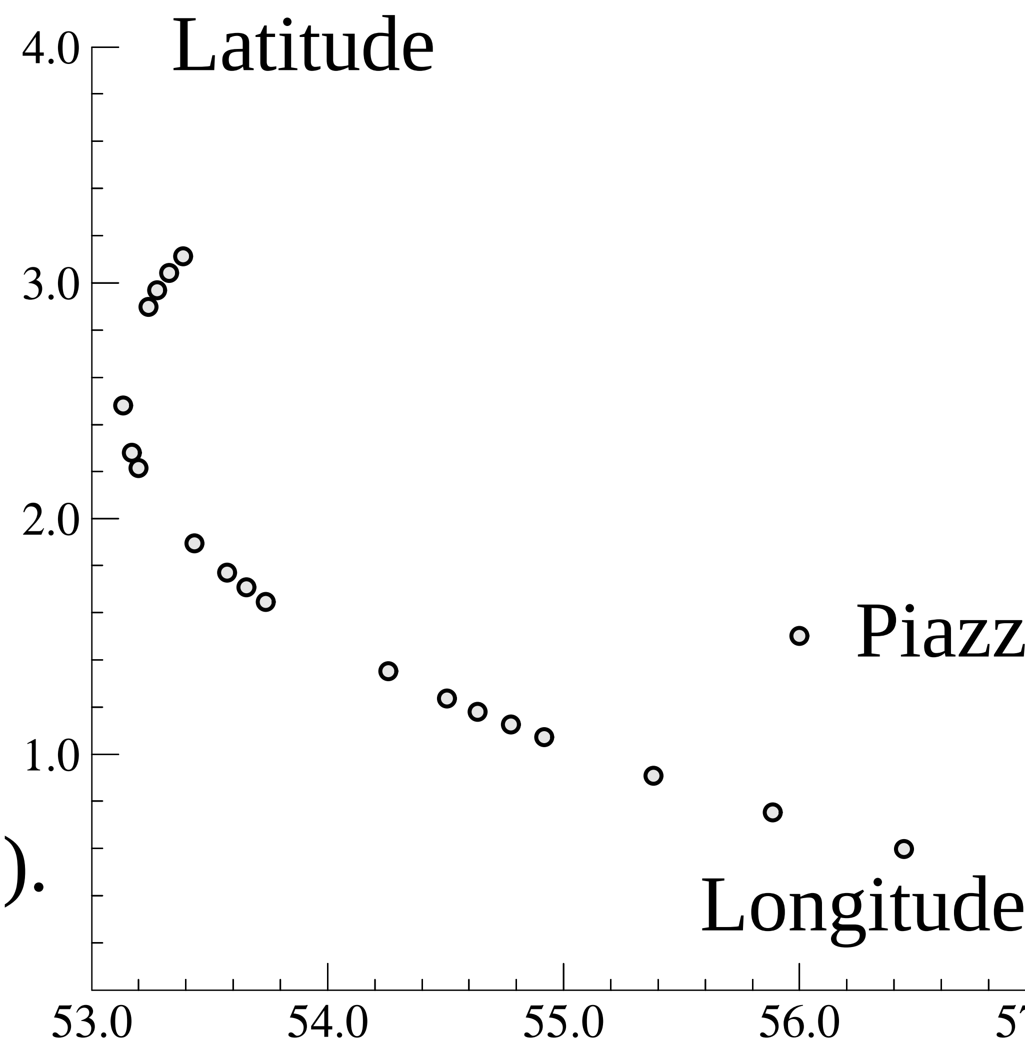

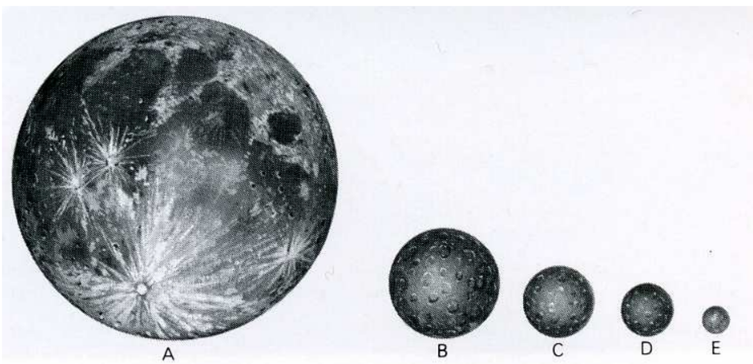

PROBLEM: Find this planet again!!

1801	Longitude	Latitude		Longitude	Latitude
Jan. 1	$53^0\ 23'\ 06.38''$	$3^0\ 06'\ 45.16''$	23	$53^0\ 44'\ 12.46''$	$1^0\ 38'\ 46.78''$
2	$53^0\ 19'\ 38.18''$	$3^0\ 02'\ 26.46''$	28	$54^0\ 15'\ 18.52''$	$1^0\ 21'\ 04.92''$
3	$53^0\ 16'\ 37.70''$	$2^0\ 58'\ 08.04''$	30	$54^0\ 30'\ 10.52''$	$1^0\ 14'\ 14.24''$
4	$53^0\ 14'\ 21.44''$	$2^0\ 53'\ 51.98''$	31	$54^0\ 38'\ 05.58''$	$1^0\ 10'\ 51.02''$
10	$53^0\ 07'\ 57.64''$	$2^0\ 28'\ 53.64''$	Feb. 1	$54^0\ 46'\ 27.14''$	$1^0\ 07'\ 34.18''$
13	$53^0\ 10'\ 05.60''$	$2^0\ 16'\ 46.08''$	2	$54^0\ 55'\ 01.52''$	$1^0\ 04'\ 18.10''$
14	$53^0\ 11'\ 54.20''$	$2^0\ 12'\ 54.02''$	5	$55^0\ 22'\ 45.20''$	$0^0\ 54'\ 34.54''$
19	$53^0\ 26'\ 01.98''$	$1^0\ 53'\ 37.82''$	8	$55^0\ 53'\ 04.52''$	$0^0\ 45'\ 08.28''$
21	$53^0\ 34'\ 22.68''$	$1^0\ 46'\ 13.06''$	11	$56^0\ 26'\ 28.20''$	$0^0\ 35'\ 55.02''$
22	$53^0\ 39'\ 11.58''$	$1^0\ 42'\ 28.80''$			

The observations of Piazzi

Many astronomers took part at the great challenge of the re-discovery (Burckhardt, Olbers, Piazzi).

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But a certain, 24 years old, “Dr. Gauss in Braunschweig” computed a different solution “nach einem eigenthümlichen Verfahren” and published it the 29th of Sept. 1801.

Still better solution in December 1801:

Sonnenferne	$326^0\ 53'\ 50''$
Ω	$81^0\ 1'\ 44''$
Neigung der Bahn	$10^0\ 36'\ 21''$
Logarithmus der halben grossen Axe	0.4414902
Excentricität	0.0819603
Epoche: 31 Dec. 1800 mittl. helioc. Länge	$77^0\ 54'\ 29''$

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The 7th of December 1801, Freiherr von Zach re-discovered Ceres precisely at the position predicted by Gauss.

How did Gauss compute his solution?

> Gewiss, jeder der die Rechnungen kennt, die die Bestimmung der Elemente eines Planeten und dann jeder daraus herzuleitende Ort erfordert, muss es bewundern, wie ein einzelner Mann in so kurzen Zeiträumen so vielfache mühsame Rechnungen zu vollenden vermögend war.
>
> (von Zach, März 1805, see Gauss *Werke* 6, p. 262)

Published posthumely; Werke vol. 11, pp. 221-252, and vol. 6, p. 199–402.

BESTIMMUNG DER BAHNEN DER HIMMELSKÖRPER.

[Aus Handbuch 16, Bb, Den astronomischen Wissenschaften gewidmet, November 1801, S. 1—8.]

[1.]

Bedeutung der hier vorkommenden Zeichen.

α ... Geocentrische Länge des Weltkörpers	t ... Zeit der Beobachtung
β ... Geocentrische Breite	v ... Länge in der Bahn
θ ... Tangente der Breite	M ... Wahre Anomalie [*)]
Δ ... Kurtirter Abstand von der Erde	E ... Eccentrische Anomalie [*)]
λ ... Heliocentrische Länge	N ... Mittlere Anomalie [*)]
β ... Heliocentrische Breite	u ... Mittlere Länge
ϑ ... Tangente dieser Breite	R ... Abstand der Sonne von der Erde
ρ ... Kurtirter Abstand von der Sonne	V ... Wahre Länge der Sonne
r ... Wahrer Abstand von der Sonne	U ... Mittlere Länge der Sonne.

Dieselben Zeichen mit Linien haben ähnliche Bedeutungen für andere achtungen.

. Länge des aufsteigenden Knoten
. Neigung der Bahn
. Tangente derselben
. Ort der Sonnenferne
. Eccentricität
. Bogen dessen Sinus $= e$
. Halbe grosse Axe
. Halbe kleine Axe
. Halber Parameter

Anm. Die Längen sind sämmtlich siderisch, oder von einem festen, übrigens willkürlichen Punkte des Himmels an gezählt.

[2.]

Formeln zur ersten Annäherung aus dreien Beobachtungen.

$$\frac{R'}{\Delta'}\left(1-\frac{R'^3}{r'^3}\right)=\frac{\theta\sin(\alpha''-\alpha')-\theta'\sin(\alpha''-\alpha)+\theta''\sin(\alpha'-\alpha)}{\frac{1}{2}(U'-U)(U''-U')\{\theta''\sin(V'-\alpha)-\theta\sin(V'-\alpha'')\}}\text{[*)]},$$

$$\text{a.]}\quad \text{genau}\left[\frac{1}{\Delta' R' R'}\left(1-\frac{R'^3}{r'^3}\right)\right]=\frac{d\alpha'\,dd\theta'-d\theta'\,dd\alpha'+\theta'\,d\alpha'^3}{dU^2\{\theta'\cos(V'-\alpha')\,d\alpha'+\sin(V'-\alpha')\,d\theta'\}}.$$

Hat man nach dieser Formel $\frac{R'}{\Delta'}\left(1-\frac{R'^3}{r'^3}\right)$ bestimmt, so findet man dar- leicht durch Verbindung mit der Gleichung

$$\frac{r'}{\Delta'}=\sqrt{\left(1+\theta'\theta'+\frac{R'R'}{\Delta'\Delta'}-2\frac{R'}{\Delta'}\cos(V'-\alpha')\right)}$$

ittelst weniger Versuche einen nahen Werth von Δ'.

$$\frac{\Delta''}{\Delta}=\frac{\theta'\sin(V'-\alpha)-\theta\sin(V'-\alpha')}{\theta''\sin(V'-\alpha')-\theta'\sin(V'-\alpha'')}\cdot\frac{t''-t'}{t'-t}=\frac{\frac{\sin(V'-\alpha)}{\sin(V'-\alpha')}-\frac{\theta}{\theta'}}{\frac{\theta''}{\theta'}-\frac{\sin(V'-\alpha'')}{\sin(V'-\alpha')}}\cdot\frac{t''-t'}{t'-t}\text{[**)]}.$$

g genau ist

$$\text{b.]}\quad \frac{2d\Delta'}{\Delta'}=-\frac{\theta'\,dd\alpha'\cos(V'-\alpha')+(\theta'\,d\alpha'^2+dd\theta')\sin(V'-\alpha')}{\theta'\,d\alpha'\cos(V'-\alpha')+d\theta'.\sin(V'-\alpha')}.$$

In Ansehung der ersten Formel ist noch zu bemerken, dass $U'-U$,

[*) Vgl. Werke VI, S. 159.]
[**) Vgl. Werke VI, S. 157.]

$U''-U'$ in Theilen des Radius ausgedrückt werden müssen; auf die ist log (mot. med. ⊙ diurn.) $= 8{,}235\,5820.792$[*)].

Nachdem man nun Δ' und $\frac{\Delta''}{\Delta}$ bestimmt hat, kann man, hinreiche nau zur ersten Näherung

$$\log\Delta=\log\Delta'-\frac{t'-t}{t''-t}\log\frac{\Delta''}{\Delta},\qquad \log\Delta''=\log\Delta'+\frac{t''-t'}{t''-t}\log\frac{\Delta''}{\Delta}$$

setzen.

Ist die Neigung der Bahn sehr gering, so sind obige Gleichunge brauchbar; die beobachtete Länge, ihre Veränderung und deren Zu Abnahme gibt sodann bloss folgende Gleichung:

$$\text{[c.]}\qquad 0=2\frac{d\Delta}{dU}\frac{d\alpha}{dU}+\Delta\frac{d^2\alpha}{dU^2}+R\left(\frac{A^3}{R^3}-\frac{A^3}{r^3}\right)\sin(V-\alpha).$$

[Im übrigen ist:]

$$\text{[d.]}\quad 0=\theta.\Delta\cdot\frac{d\alpha^2}{dU^2}+2\frac{d\Delta}{dU}\frac{d\theta}{dU}+\Delta\frac{d^2\theta}{dU^2}-R\theta\left(\frac{A^3}{R^3}-\frac{A^3}{r^3}\right)\cos(V-\alpha).$$

[S. 3]

[3.]

Vorschriften zur Berechnung der Elemente, aus zweien geoc schen Örtern, der Zwischenzeit, und den zugehörigen Abstände

$$\rho=\sqrt{\{RR+\Delta\Delta-2R\Delta\cos(V-\alpha)\}}.$$

Ganz allgemein

$$\begin{cases}\Delta\sin\alpha-R\sin V=\rho\sin\lambda\\ \Delta\cos\alpha-R\cos V=\rho\cos\lambda\end{cases};$$

folglich

$$\text{I.}\begin{cases}\Delta\sin(V-\alpha)=\rho\sin(V-\lambda)\\ \Delta\cos(V-\alpha)-R=\rho\cos(V-\lambda)\end{cases},\qquad \text{II.}\begin{cases}R\sin(V-\alpha)=\rho\sin(\alpha-\lambda)\\ R\cos(V-\alpha)-\Delta=-\rho\cos(\end{cases}$$

$$\text{III.}\begin{cases}(\Delta+R)\sin\frac{1}{2}(V-\alpha)=\rho\sin\{\frac{1}{2}(V+\alpha)-\lambda\}\\ (\Delta-R)\cos\frac{1}{2}(V-\alpha)=\rho\cos\{\frac{1}{2}(V+\alpha)-\lambda\}\end{cases}$$

[*)] Wenn man auf die Erdmasse mit Rücksicht nimmt, muss man log mot Planetae in distantia media setzen: $8{,}235\,5814.21\cdot[A^{\frac{3}{2}}:a^{\frac{3}{2}}$, wo wie auch im Fo A die halbe große Achse der Erdbahn bedeutet].

[**) Vgl. Theoria motus, art. 124, Werke VII, 1906, S. 165.]

$\ldots),\quad \frac{e}{p}\cos\left(\frac{1}{2}(v''+v)-\omega\right)=\dfrac{\frac{1}{p}-\frac{1}{2}\left(\frac{1}{r''}+\frac{1}{r}\right)}{\cos\frac{1}{2}(v''-v)},$

$\ldots\ \frac{p}{\ldots s\,\varphi},\quad a=\frac{p}{\cos\varphi^2},$

$\text{med.}=\dfrac{\text{mot. diurn.}\odot\text{med.}}{a^{\frac{3}{2}}},$

$\ldots -\omega,\quad M''=v''-\omega,$

$\sqrt{\frac{r}{p}}\}$ zur Probe $\quad \operatorname{tg}\frac{1}{2}E=\operatorname{tg}\frac{1}{2}M\ \operatorname{tg}\left(\frac{1}{2}\varphi+45^0\right),$

$\sqrt{\frac{r''}{p}}\}\quad \operatorname{tg}\frac{1}{2}E''=\operatorname{tg}\frac{1}{2}M''\ \operatorname{tg}\left(\frac{1}{2}\varphi+45^0\right),$

$\ldots E''=\sin\varphi\sin M''\cdot\frac{r''}{b},\quad t''-t=\left.\frac{N''-N}{\text{mot. diurn. med.}}\right]$

…693	$[\frac{1}{2}(rr+r''r'')\ldots]$	0,856 1994
…071	$\left[\frac{U''-U}{v''-v}\ldots\right]$	0,638 3472
…764	$[(\sqrt{p})\ldots]$	0,217 8522
…292	$[(p)\ldots]$	0,435 7044
…472	$\left[\sqrt[3]{\left(\frac{2\mathfrak{R}\mathfrak{R}}{rr+r''r''}\right)^4}\ldots\right]$	$-$1223 [*)]
	Parameter $[p\ldots]$	0,435 5821

$\ldots n\frac{1}{4}(v''-v)^2\ldots]$ 7,514 4812 $\qquad \left[\frac{1}{2}\left(\frac{1}{r}+\frac{1}{r''}\right)-\frac{1}{p}\ldots\right]$ 7,805 6761

$\ldots os\frac{1}{2}(v''-v)\ldots]$ 9,998 5777 $\qquad \left[\frac{2\sin\frac{1}{4}(v''-v)^2}{\cos\frac{1}{2}(v''-v)}\ldots\right]$ 7,515 9035

[Diff.] 7,515 9035 $\qquad$ [Summe $\ast$] 5,321 58

$[\mathfrak{R}\ldots]$ 0,428 05386

$[\mathfrak{R}\mathfrak{R}\ldots]$ 0,856 1077

…8 $\quad [\frac{1}{2}(rr+r''r'')\ldots$ 0,856 1]994

…9 $\quad \left[\frac{1}{2}\frac{rr+r''r''}{\mathfrak{R}\mathfrak{R}}\ldots\right.$ 0,000 0]917

…7 $\quad \left[\sqrt[3]{\left(\frac{1}{2}\frac{rr+r''r''}{\mathfrak{R}\mathfrak{R}}\right)}\ldots\right.$ 0,000 0]306

$\left[\sqrt[3]{\left(\frac{1}{2}\frac{rr+r''r''}{\mathfrak{R}\mathfrak{R}}\right)^4}\ldots\right.$ 0,000 1223]

…ehenden letzten Annäherung zur Bestimmung von p; auch in … vorhergehenden Annäherungen, aufgezeichnet; der zuerst ge… eichnet.]

$\left[\frac{1}{2}\left(\frac{1}{r''}-\frac{1}{r}\right)=\right]$ 0,002 140 0125

$[\ldots]$ 7,330 4163

$[\sin\frac{1}{2}(v''-v)\ldots]$ 8,907 4003

$\left[\frac{e}{p}\sin(\frac{1}{2}(v''+v)-\omega)\ldots\right]$ 8,423 0160

$[\sin(\frac{1}{2}(v''+v)-\omega)\ldots]$ 9,987 6267

$\left[\frac{e}{p}\ldots\right]$ 8,435 3893

$[p\ldots$ 0,]435 5821

$[e=\sin\varphi\ldots]$ 8,870 9714

$[\frac{1}{2}\sin\varphi\ldots]$ 8,569 9414

$[\cos\frac{1}{2}\varphi\ldots]$ 9,999 6997 $\qquad \varphi=4^0\,15'\,39''00$

$\log\sin\frac{1}{2}\varphi=8{,}570\,2417$ [*)] $\qquad \frac{1}{2}\varphi=2\quad 7\,49''50$

$\left[\frac{1}{2}\left(\frac{1}{r''}+\frac{1}{r}\right)-\frac{1}{p}\right.=$ 0,00]6 392 5795

$[\ldots]$ 7,805 6761

$[\cos\frac{1}{2}(v''-v)\ldots]$ 9,998 5777

$\frac{e}{p}\cos(\frac{1}{2}(v''+v)-\omega)\ldots$ 7,807 0984 [n]

$\frac{e}{p}\sin(\frac{1}{2}(v''+v)-\omega)\ldots$ 8,423 0160

$[\operatorname{cotang}(\frac{1}{2}(v''+v)-\omega)\ldots]$ 9,384 0824 [n]

$[\frac{1}{2}(v''+v)-\omega=]$	$103^0\,36'\,43''53$
$[\frac{1}{2}(v''+v)=]$	73 27 55″20
$[\omega=]$	329 51 11″67
$[\frac{1}{2}(v''+v)-\omega=]$	$103^0\,36'\,43''53$
$[\frac{1}{2}(v''-v)=]$	4 38 3″96
$[M=v-\omega]$	98 58 39″57
$[M''=v''-\omega]$	108 14 47″49
$p\ldots$	0,435 5821
$b\ldots$	0,436 7841
$a\ldots$	0,437 9861
$[\sqrt{a}\ldots]$	0,218 9930
$[a^{3/2}\ldots]$	0,656 9791
[mot. diurn. $\odot$ med.	3,]550 0071
[mot. diurn. med.]	2,893 0279

[Zur Probe von E und E'']

[S. 3 u. 10] $\quad [\sin M\ldots]$ 9,994 6467

$[\sin\frac{1}{2}\varphi\ldots]$ 8,570 2417

[Summe] 8,564 8884

$\left[\sqrt{\frac{r}{p}}\ldots\right]$ 25031

$[\sin\frac{1}{2}(E-M)=]$ 8,562 3853

$[\frac{1}{2}(E-M)=2^0\,5'\,31''94$

$[\sin M''\ldots]$ 9,977 5947

$[\sin\frac{1}{2}\varphi\ldots]$ 8,570 2417

[Summe] 8,547 8364

$\left[\sqrt{\frac{r''}{p}}\ldots\right]$ $-$ 49936

$[\sin\frac{1}{2}(E''-M'')]$ 8,542 8428

$[\frac{1}{2}(E''-M'')=]\ 2^0\,0'\,0''39$

$[\frac{1}{2}M=]$	$49^0\,29'\,19''785$
$[\operatorname{tang}(\frac{1}{2}\varphi+45^0)\ldots]$	0,032 3264
$[\operatorname{tang}\frac{1}{2}M\ldots]$	0,068 3297
$[\operatorname{tang}\frac{1}{2}E\ldots]$	0,100 6561
$[\frac{1}{2}M''=]$	$54^0\,7'\,23''745$
$[\operatorname{tang}(\frac{1}{2}\varphi+45^0)\ldots]$	0,032 3264
$[\operatorname{tang}\frac{1}{2}M''\ldots]$	0,140 7051
$[\operatorname{tang}\frac{1}{2}E''\ldots]$	0,173 0315

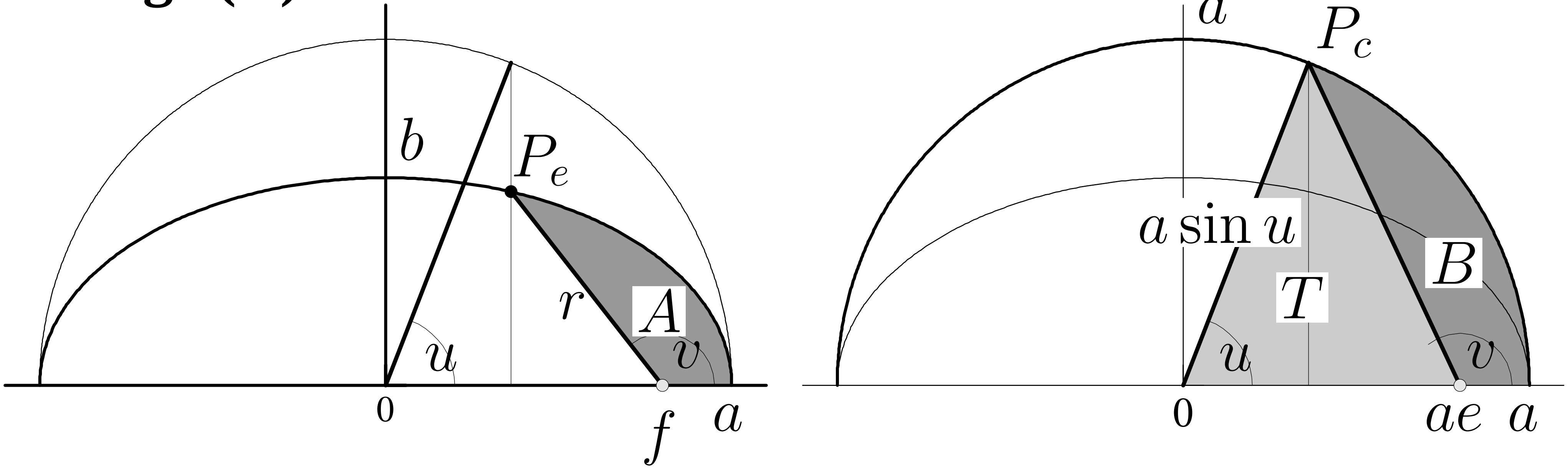
[*) Anscheinend nach der Formel $\sin\frac{1}{2}\varphi=\frac{1}{2}\frac{\sin\varphi}{\cos\frac{1}{2}\varphi}$ gerechnet.]

All the difficulty stems from the great number of variables:

Elements of orbit		(A)	Heliocentric coordinates	(B)	Geocentric spherical coordinates
w	arg. of perihelion				
Ω	long. of ascend. node				
i	inclination of orbit	$\Longleftrightarrow$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$\Longleftrightarrow$	$\begin{pmatrix} \rho \\ \lambda \\ \beta \end{pmatrix}$
a	semi-major axis				
e	eccentricity				
l_0	mean heliocent. long.				

The quantities measured are the angles λ and β (the distance ρ is unknown, of course) for several time values, the quantities to be computed are the elements of the orbit. So we need formulas for the connecting passages (A) and (B).

Passage (A).



For given t, must find u.
Kepler's second law ('same times, same areas')

$$\Rightarrow \qquad \frac{A}{ab\pi} = \frac{t}{P}. \quad \Rightarrow \quad nt = u - e\sin u, \qquad n = \frac{2\pi}{P}$$

(Kepler's equation) since $B = \frac{a}{b}A$, $B = \frac{a^2}{2}(u - e\sin u)$.

Kepler's third law: n^2a^3 is a known constant.

$\Rightarrow$ (using spherical trigonometry) coordinates (x, y, z).

Passage (B). For this, we have to know the solar geocentric coordinates (X, Y, Z) (again by Kepler's laws, this time applied to the earth's orbit) and we obtain the geocentric ecliptic coordinates of the planet by adding these and taking spherical coordinates

$$\begin{aligned} \xi &= x + X &= \rho \cos \beta \cos \lambda \\ \nu &= y + Y &= \rho \cos \beta \sin \lambda \\ \zeta &= z + Z &= \rho \sin \beta. \end{aligned}$$

Gauss' Procedure. At that time, it was "easy" to solve

$$\begin{pmatrix} \rho_1 \\ \lambda_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \rho_2 \\ \lambda_2 \\ \beta_2 \end{pmatrix} \overset{\text{(B)}}{\Rightarrow} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \overset{\text{(A)}}{\Rightarrow} \begin{pmatrix} w \\ \Omega \\ i \\ a \\ e \\ l_0 \end{pmatrix}$$

However, ρ_1 and ρ_2 are unknown!

Gauss: very complicated formula manipulations $\Rightarrow$ compute

$$\begin{pmatrix} \lambda_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \lambda_2 \\ \beta_2 \end{pmatrix} \overset{\text{(B)}}{\Rightarrow} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda_2 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \lambda_3 \\ \beta_3 \end{pmatrix} \overset{\text{(B)}}{\Rightarrow} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Thereby, it was advantageous to have t_2 exactly in the middle.
Gauss started with the data

Jan. 2, Jan. 22, and Feb. 11.

Recomputed repeatedly by changing dates and data.

The Method of Least Squares

> hat man schon Beobachtungen von 1 oder mehrern Jahren . . ., so halte ich den Gebrauch der Differential-Änderung, wobei man eine beliebige Zahl von Beobachtungen zum Grunde legen kann, für das beste Mittel.
>
> (Gauss, *Summarische Übersicht*, publ. 1809)

Ceres rediscovered Dec. 1801; $\Rightarrow$ much more data. Start of use of the Method of Least Squares; no publication.

1805: Legendre publishes Nouvelles méthodes pour la determination des orbites des comètes with appendix méthode des moindres quarrés.

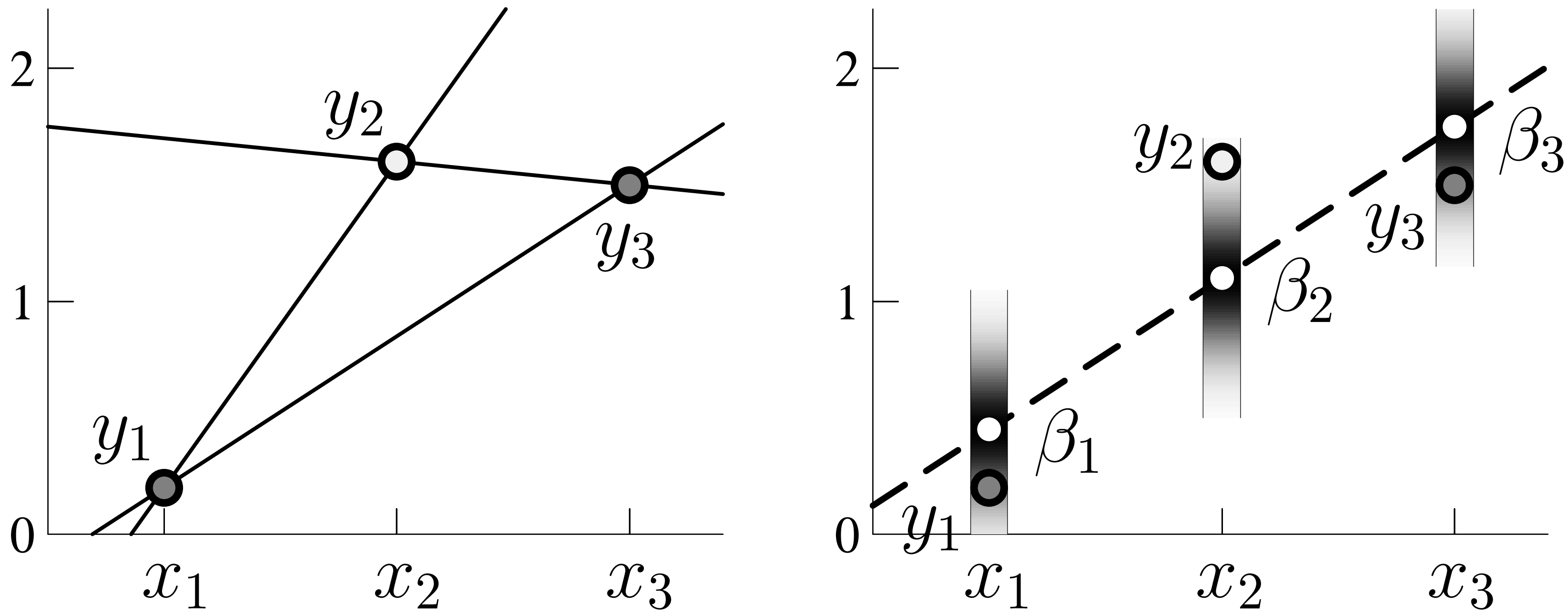
1809: Gauss publishes Theoria motus corporum celestum containing Principium nostrum "which I have made use of since 1795".

> je n'ai jamais appellé *principium nostrum* un principe qu'un autre avait publié avant moi..
>
> (Legendre in a letter to Gauss, without answer.)

Gauss' Probabilistic Justif. of the Least Squares Principle.

To explain the idea, we treat a simple problem, i.e., the approximation of three 'observations' x_i, y_i $(i = 1, 2, 3)$ by an 'orbit' which is a straight line

$$y = a + bx \qquad \Rightarrow \qquad \beta_i = a + bx_i$$



mesures y_i are random samplings.

Probability for having measured y_i (to a precision of Δy):

$$P(0 \leq \beta_i - y_i \leq \Delta y) = \frac{e^{-\frac{(\beta_i - y_i)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \Delta y.$$

Now, the probability for having measures the *three* values y_1, y_2, y_3 (to a precision of Δy) is the product, i.e.,

$$\left(\frac{\Delta y}{\sigma\sqrt{2\pi}}\right)^3 \prod_{i=1}^{3} e^{-\frac{(\beta_i - y_i)^2}{2\sigma^2}} = \left(\frac{\Delta y}{\sigma\sqrt{2\pi}}\right)^3 e^{-\frac{\sum_{i=1}^{3}(\beta_i - y_i)^2}{2\sigma^2}}.$$

We have then *maximum likelihood* of our result, when this probability is maximal, i.e., when the exponent

$$\sum_{i=1}^{3} (\beta_i - y_i)^2 = \text{min !} \qquad = \text{principium nostrum!}$$

Gauss' "Normal Equations".

For our example we have $\sum_{i=1}^{3}(a + bx_i - y_i)^2 = \min$!

Differentiating with respect to a and b we obtain

$$\begin{pmatrix} \Sigma\, 1 & \Sigma\, x_i \\ \Sigma\, x_i & \Sigma\, x_i^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \Sigma\, y_i \\ \Sigma\, x_i y_i \end{pmatrix} \qquad \text{(normal equations)}$$

or

$$A^T A \alpha = A^T y, \quad A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{pmatrix}.$$

Good luck, that the *principium* with best probabilistic justification also leads to the easiest problem, a *linear* system.

Further Developments.

- *Gaussian elimination*. In order to prove the solvability of the normal equations, Gauss made in 1809 the first clear description of the elimination algorithm for linear equations.

- *Gauss-Newton method*. In the same paper, Gauss also explained how *nonlinear* least squares problems are linearized in the neighbourhood of a first approximate solution, which is then iteratively refined.

- *Laplace's central limit theorem*. In 1809, Laplace published his central limit theorem, showing that *any* probability function, after taking arithmetic means, tends to the normal distribution for $n \to \infty$. Soon after, he extended this to justify the principle of least squares for arbitrary probability functions and $n \to \infty$. A great publication of all these results was *Théorie analytique des probabilités* from 1812.

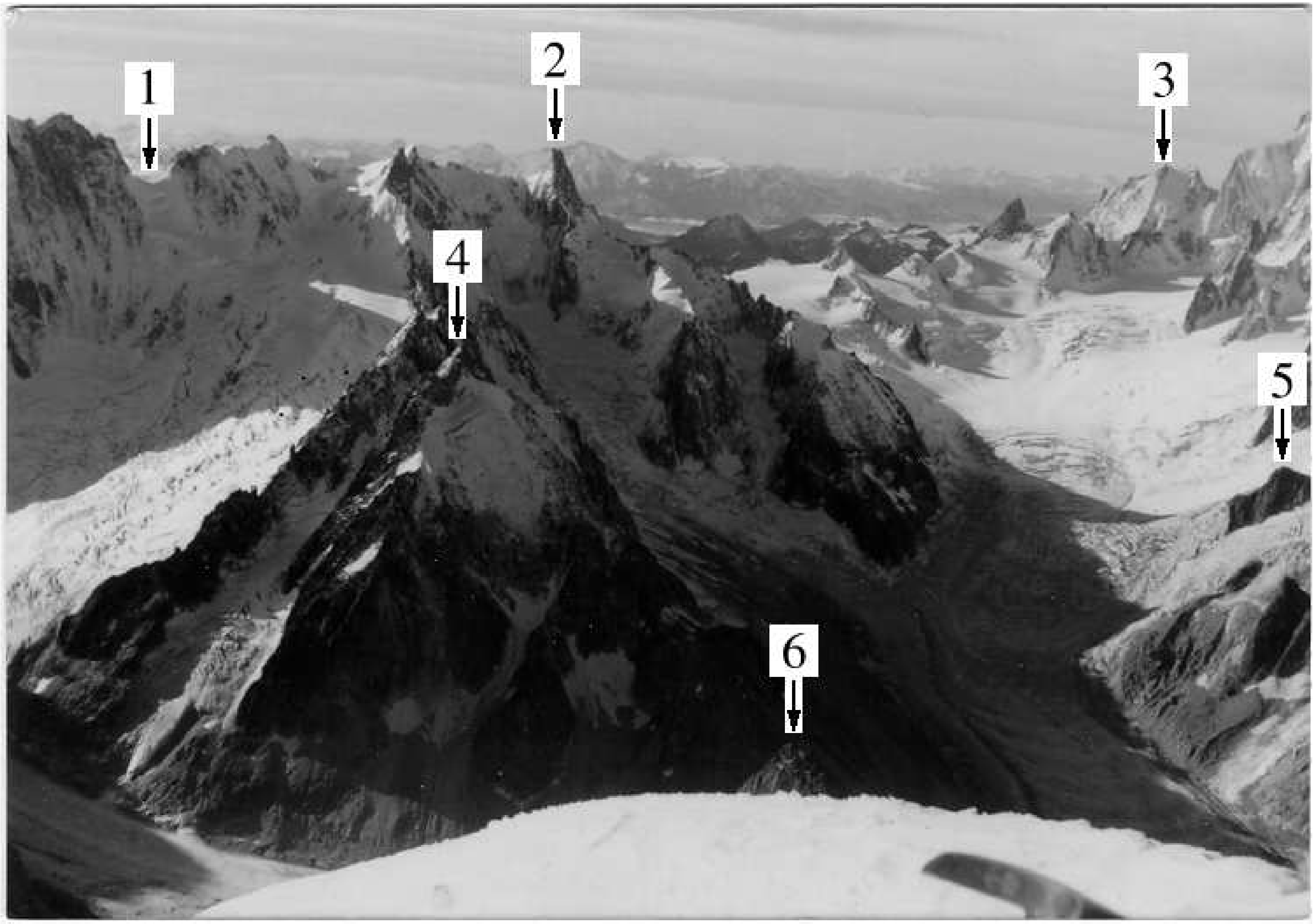
- In 1823, Gauss publishes a second fundamental treatise on least squares, *Theoria combinationis observationum erronibus minimis obnoxiae* in two parts, which contains a new justification of the least squares principle, independent of the probability function, which is today called the *Gauss-Markov Theorem*.

- In 1828, Gauss publishes a *Supplementum*, which contains impressive calculations for the geodesic triangulations of the Netherlands and the country of Hannover.

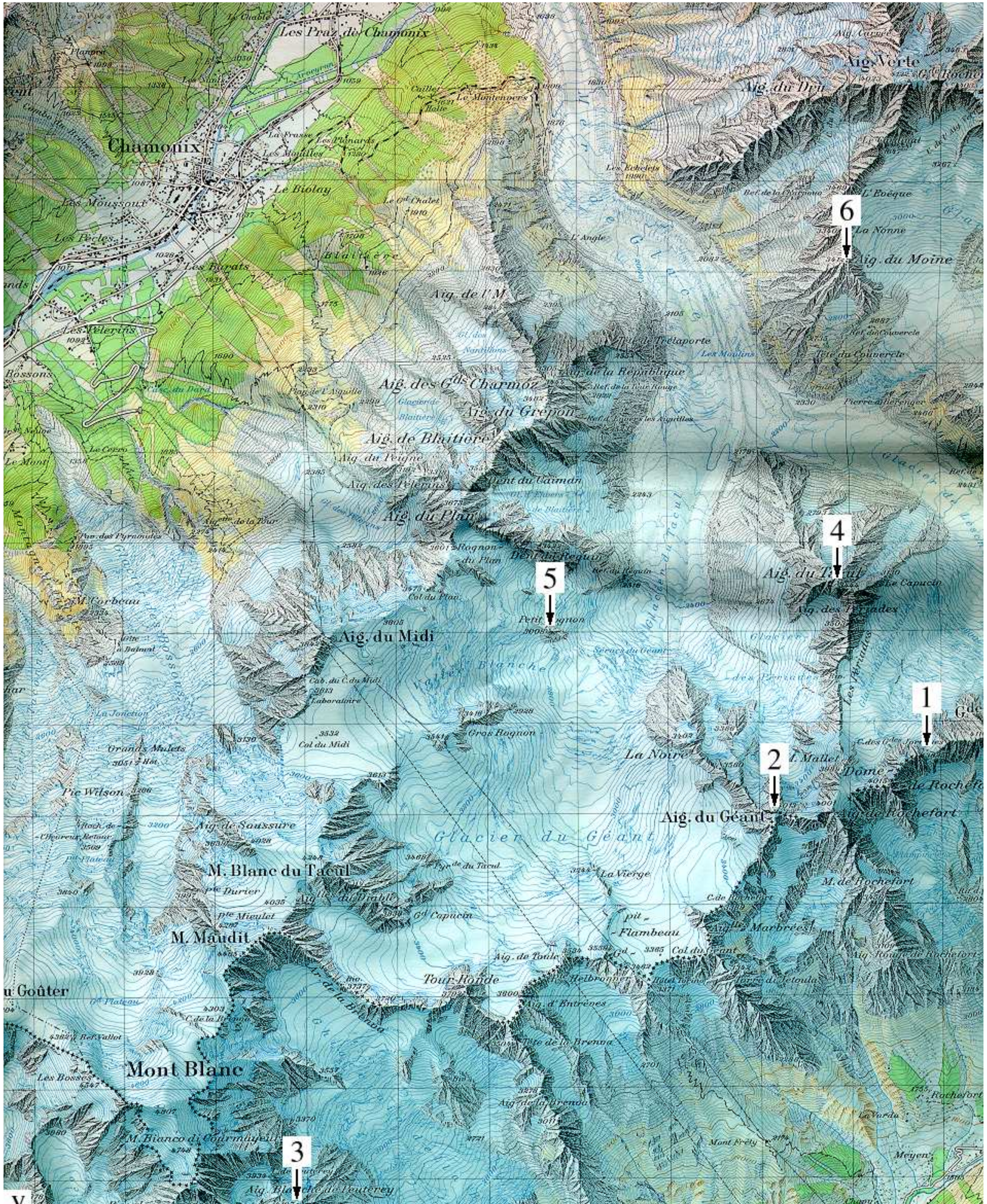
- Also in 1828, Bessel discovers, originally for the discrete case, the relation between the least squares idea, the orthogonality relations, and the Euler-Fourier formulas for the trigonometric approximation; this discovery, extended by Gram (1883) to the continuous case, is the basis of the L^2 Hilbert space theory of *Fourier series*.

- In 1845, Jacobi publishes his method for solving the normal equations with the help of successive rotations in R^2. These rotations lead in the 1950ies to Givens' method for triangularization and the first stable *eigenvalue algorithm*.
- In 1900 appears the classical paper of Karl Pearson, which combines the least squares method with the χ^2 *distribution* and led to the famous χ^2-*test for the reliability of hypotheses*.
- In 1958 appears Householder's reflection algorithm, which, by replacing Givens' rotations, leads to the *QR decomposition*, and, by Golub (1965), became the nowadays standard algorithm for least squares problems.

Example: Find the position of the camera!



A photograph (from the Montblanc region)



Chamonix
Les Praz de Chamonix
Le Biolay
Les Mousoux
Les Pèlerins
Bossons
Aig. Verte
Aig. du Dru
L'Evêque
La Nonne
Aig. du Moine
Aig. de l'M
Aig. des Gds Charmoz
Aig. du Grépon
Aig. de Blaitière
Aig. du Plan
Aig. du Midi
Petit Rognon
Aig. du Tacul
Aig. des Periades
Gros Rognon
La Noire
Aig. du Géant
Aig. de Rochefort
Glacier du Géant
M. Blanc du Tacul
M. Maudit
Mont Blanc
Tour Ronde
Col du Géant
Aig. de Saussure
Pic Wilson
Grands Mulets
M. Corbeau
Mont Fréty
Aig. Blanche de Peuterey
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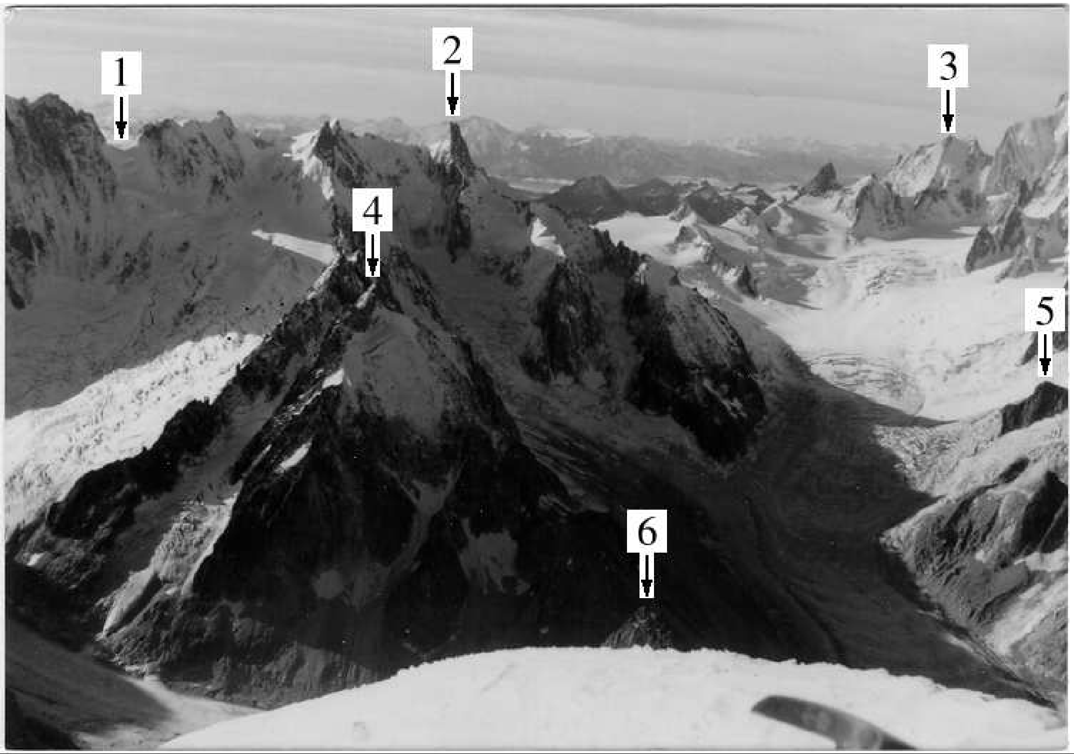


k	$\widehat{u}_k$	$\widehat{v}_k$	x_k	y_k	z_k
1. Col des Grandes Jorasses	-0.0480	0.0290	9855	5680	3825
2. Aiguille du Géant	-0.0100	0.0305	8170	5020	4013
3. Aig. Blanche de Peuterey	0.0490	0.0285	2885	730	4107
4. Aiguille du Tacul	-0.0190	0.0115	8900	7530	3444
5. Petit Rognon	0.0600	-0.0005	5700	7025	3008
6. Aiguille du Moine	0.0125	-0.0270	8980	11120	3412

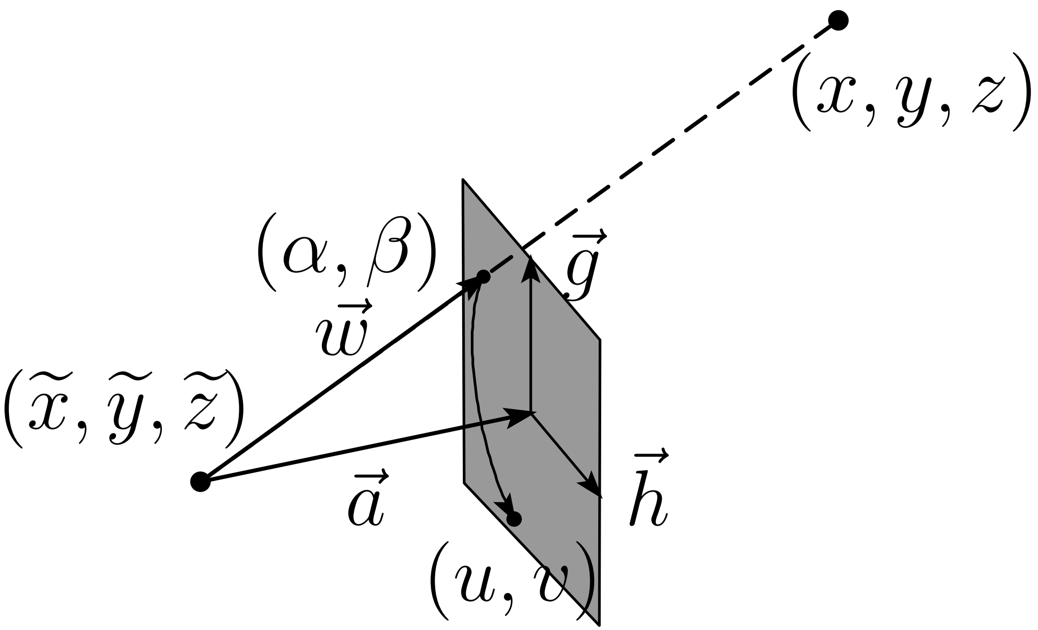
Solution of our Problem.

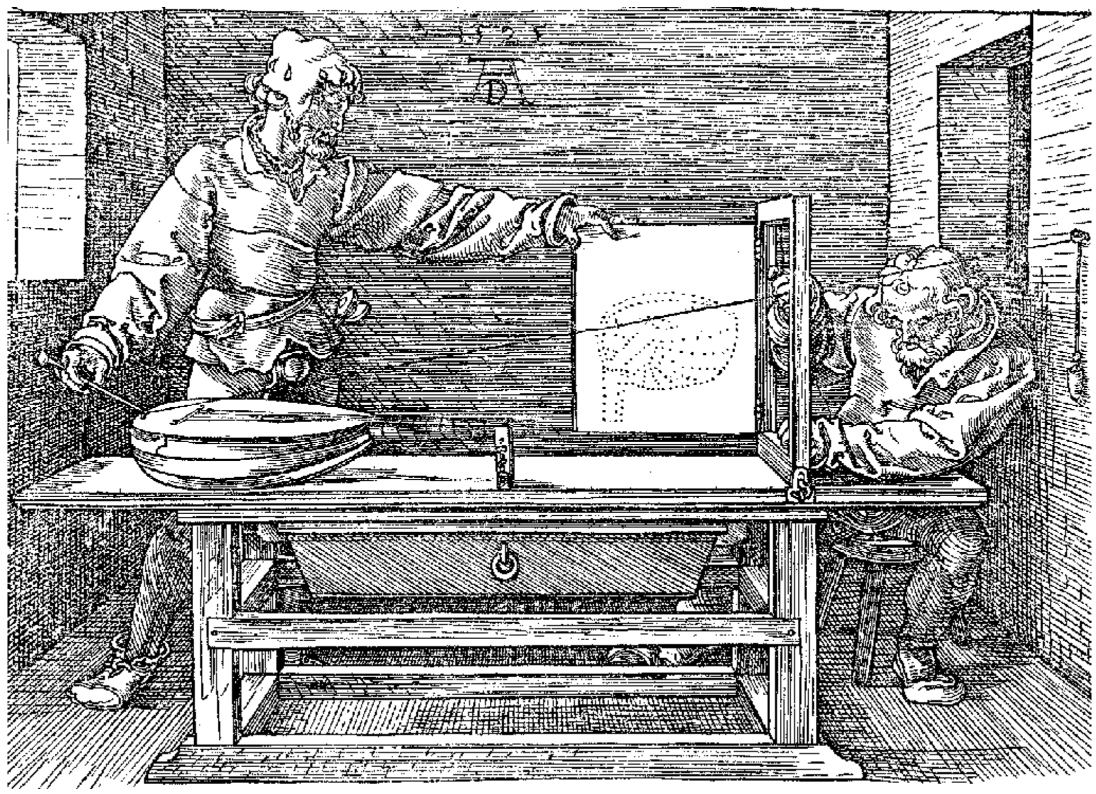
$(\widetilde{x}, \widetilde{y}, \widetilde{z})$ the position of the camera's objective

$\vec{a} = (a, b, c)$ the vector from objective to projection plane.

Rotation angle θ.

There are thus seven unknowns to determine.





Need formulas for $(x, y, z) \mapsto (u, v)$.

Develop Formulas: Orthogonal vectors in the projection plane

$$\vec{h} = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}, \quad \vec{g} = \frac{1}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}.$$

Then, for a given point (x, y, z) we compute a vector $\vec{w}$ by

$$\vec{w} = \lambda \cdot \begin{pmatrix} x - \widetilde{x} \\ y - \widetilde{y} \\ z - \widetilde{z} \end{pmatrix}$$

where the factor λ is determined by $\langle \vec{w} - \vec{a}, \vec{a} \rangle = 0$. Then $\alpha = \langle \vec{w}, \vec{h} \rangle$ and $\beta = \langle \vec{w}, \vec{g} \rangle$ are the coordinates of the projection point, which are finally rotated by θ:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

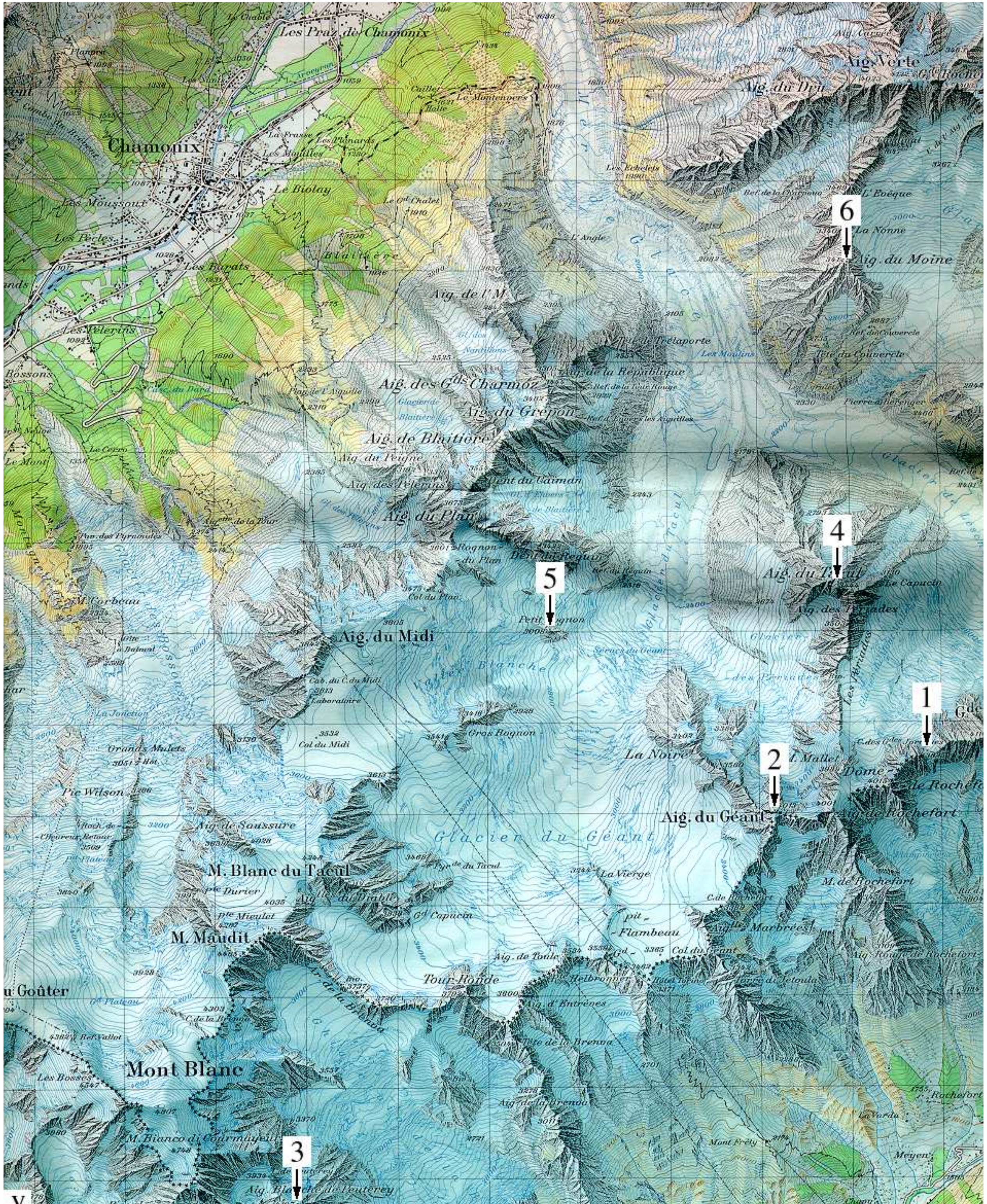
Principium Nostrum:

We have *then* the best solution, when these projected points (u_k, v_k), for the data (x_k, y_k, z_k), correspond in the best possible way to the measured data points $(\widehat{u}_k, \widehat{v}_k)$ of the photograph. Thus,

$$\sum_{k=1}^{6} \left((u_k - \widehat{u}_k)^2 + (v_k - \widehat{v}_k)^2 \right) = \min \ !$$

We need:

- A subroutine computing $f_i = (u_k - \widehat{u}_k)$, $f_{n+i} = (v_k - \widehat{v}_k)$ as function of the parameters;
- A general code which computes the partial derivatives and the solution of the normal equations;
- $\Rightarrow$ Solution: $\quad \widetilde{x} = 9679 \quad \widetilde{y} = 13139 \quad \widetilde{z} = 4131.$



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Aig. du Grépon
Aig. de Blaitière
Aig. du Peigne
Aig. des Pèlerins
Aig. du Plan
Aig. du Midi
Col du Midi
Gros Rognon
La Noire
Aig. du Géant
Aig. de Rochefort
M. de Rochefort
Glacier du Géant
Aig. de Saussure
M. Blanc du Tacul
Gd Capucin
La Vierge
Flambeau
Col du Géant
Aig. Marbrées
M. Maudit
Tour Ronde
Aig. de Toule
Mont Blanc
Les Bosses
M. Bianco di Courmayeur
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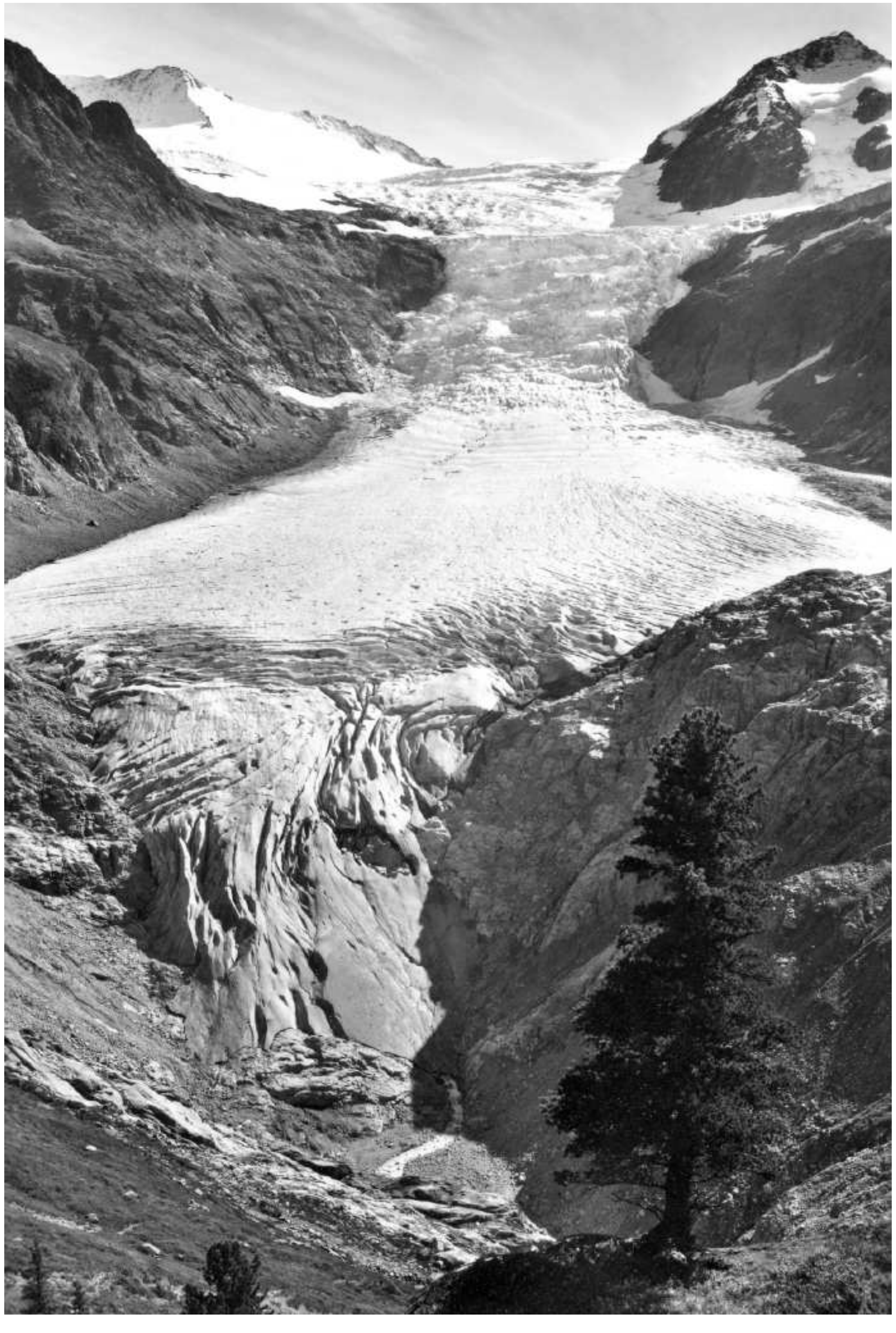
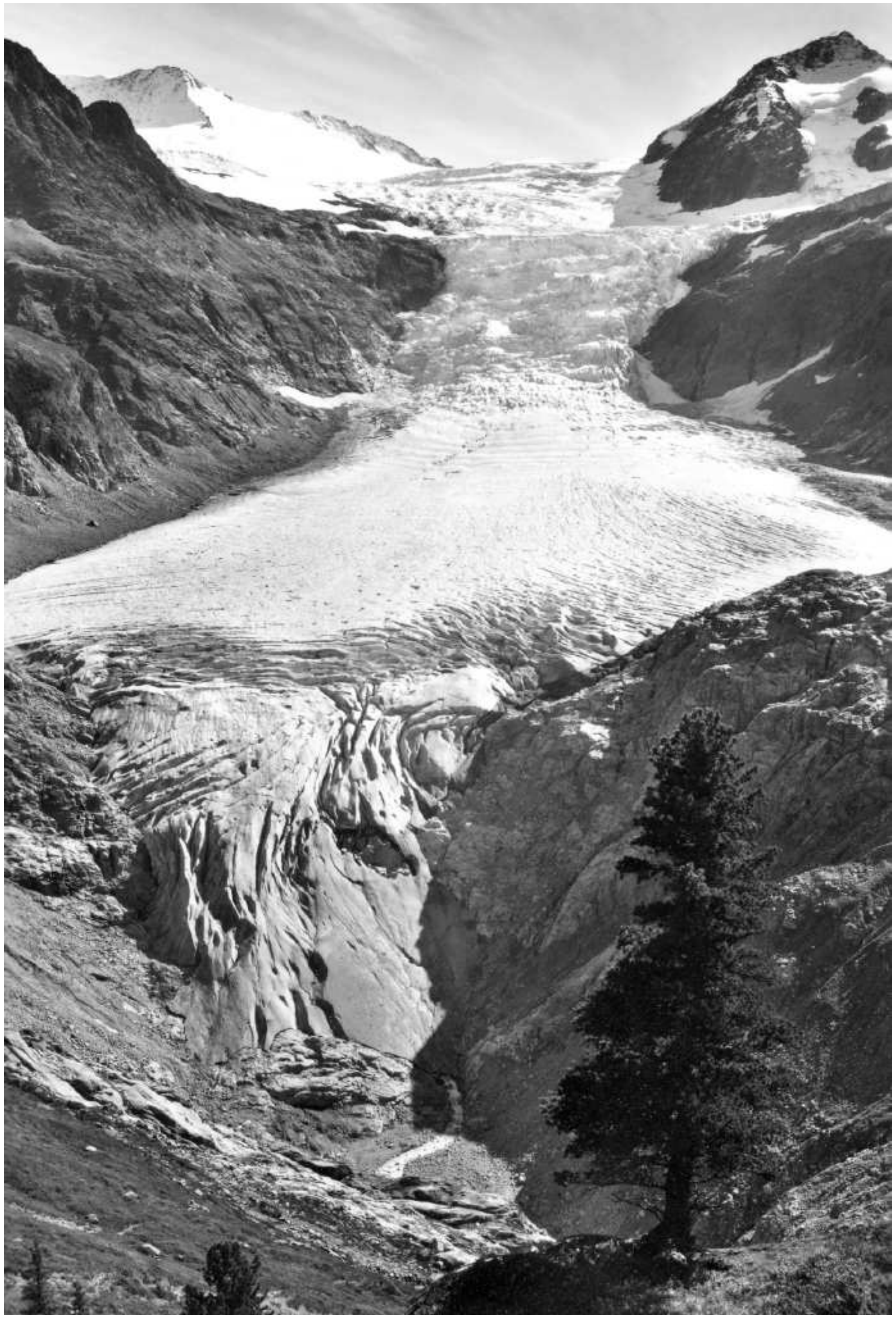
Triftgletscher 1948

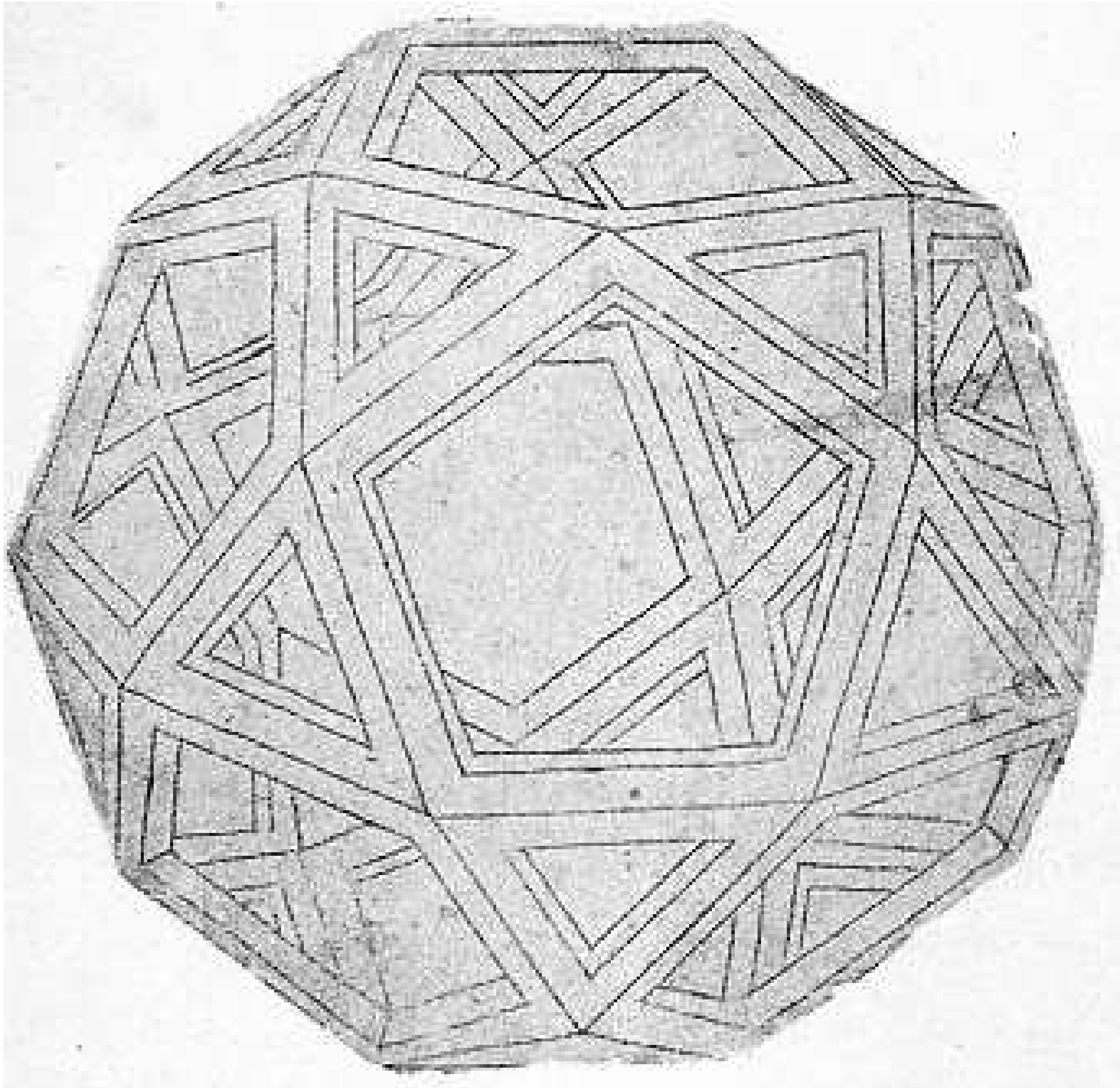


Triftgletscher 1948



Triftgletscher 2004

“Correcting” Leonardo da Vinci.



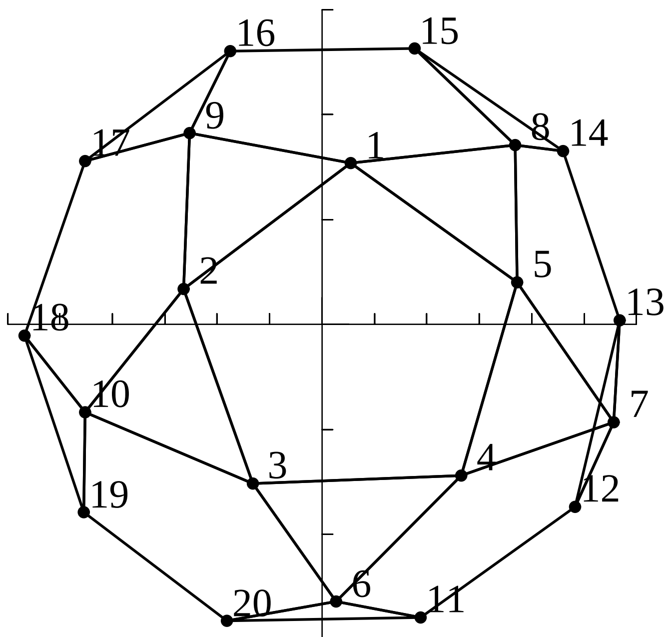
Drawing of Leonardo da Vinci (1510, Codex Atlanticus fol. 707r; Bibliotheca Ambrosiana, Milano)

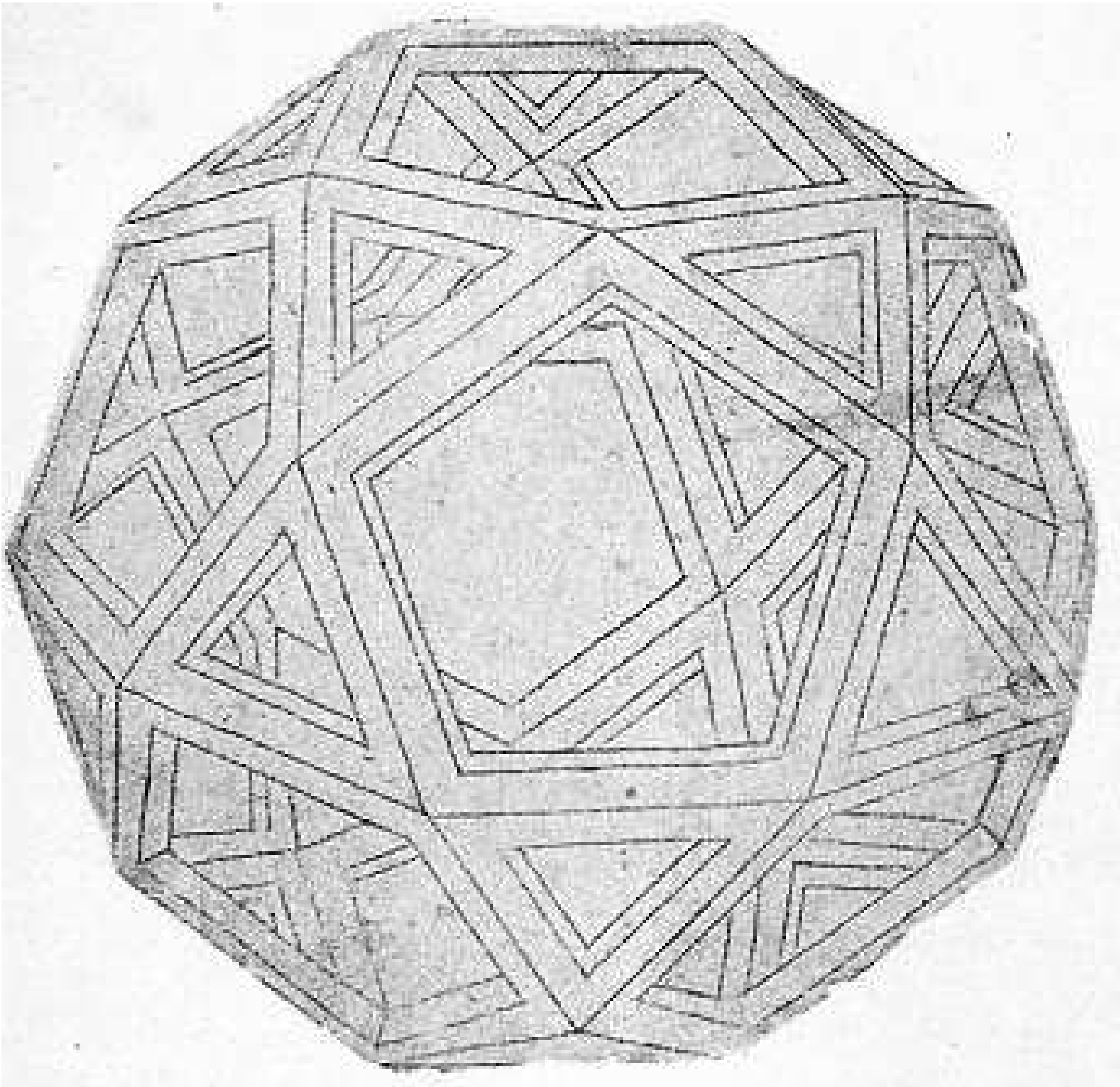
Is the drawing correct?

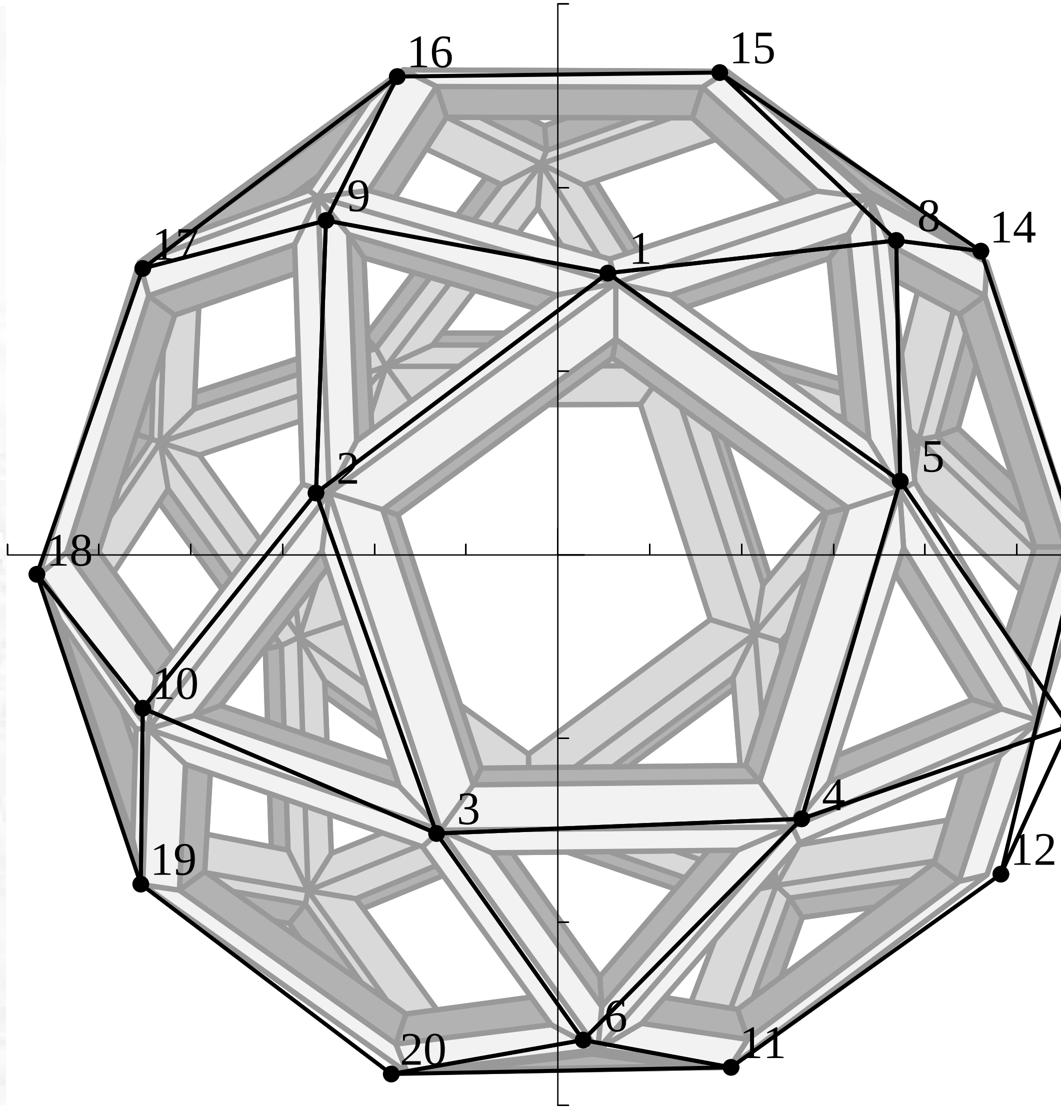
k	u_k	v_k
1	5.409	30.691
2	-26.388	6.720
3	-13.259	-30.369
4	26.517	-28.782
5	37.265	8.054
6	2.734	-52.888
7	55.650	-18.639

k	u_k	v_k
8	36.865	34.219
9	-25.283	36.394
10	-45.244	-16.728
11	18.814	-55.828
12	48.271	-34.749
13	56.767	0.764
14	46.037	33.043

k	u_k	v_k
15	17.609	52.536
16	-17.522	52.122
17	-45.244	31.161
18	-56.768	-2.147
19	-45.433	-35.867
20	-18.198	-56.563

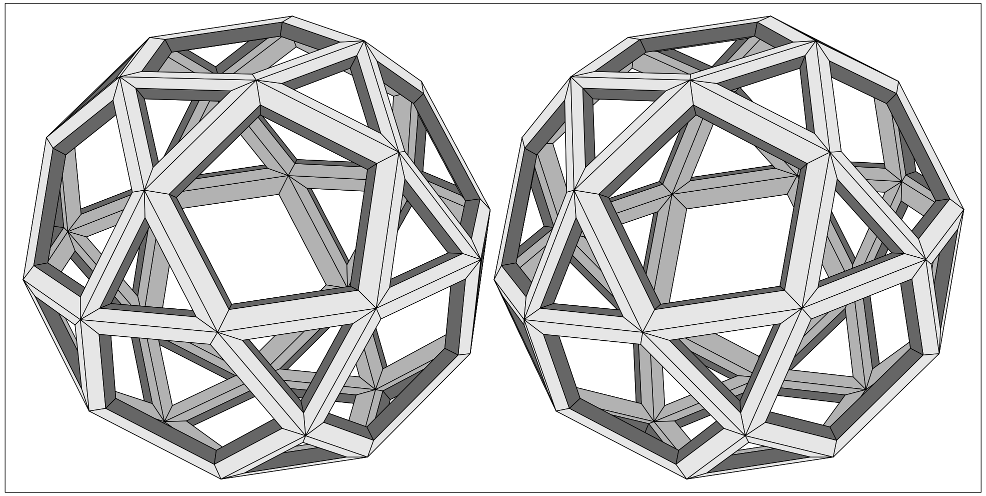






Left: Drawing of Leonardo da Vinci (1510, Codex Atlanticus fol. 707r; Bibliotheca Ambrosiana, Milano); right: Leonardo's vertices and, in grey, the 'corrected' drawing (Assyrus Abdullus & Gerhardus Wannerus, linguæ programmatoriæ Fortranus & Postscriptus, Calculatores SunBlade 100, Universitas Genavæ)

... and for the pleasure of the eye in stereo:



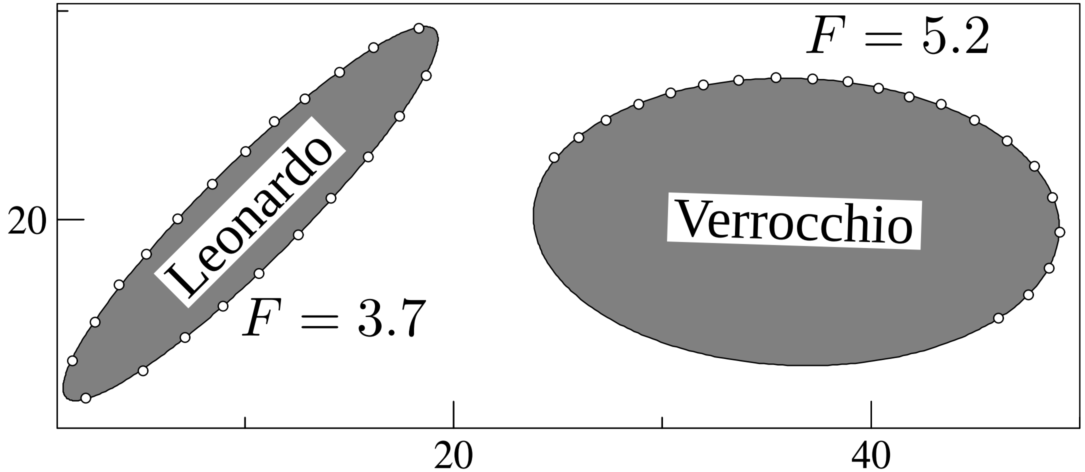
Leonardo against Verrocchio.

Dispirited is the pupil who does not surpass his master.

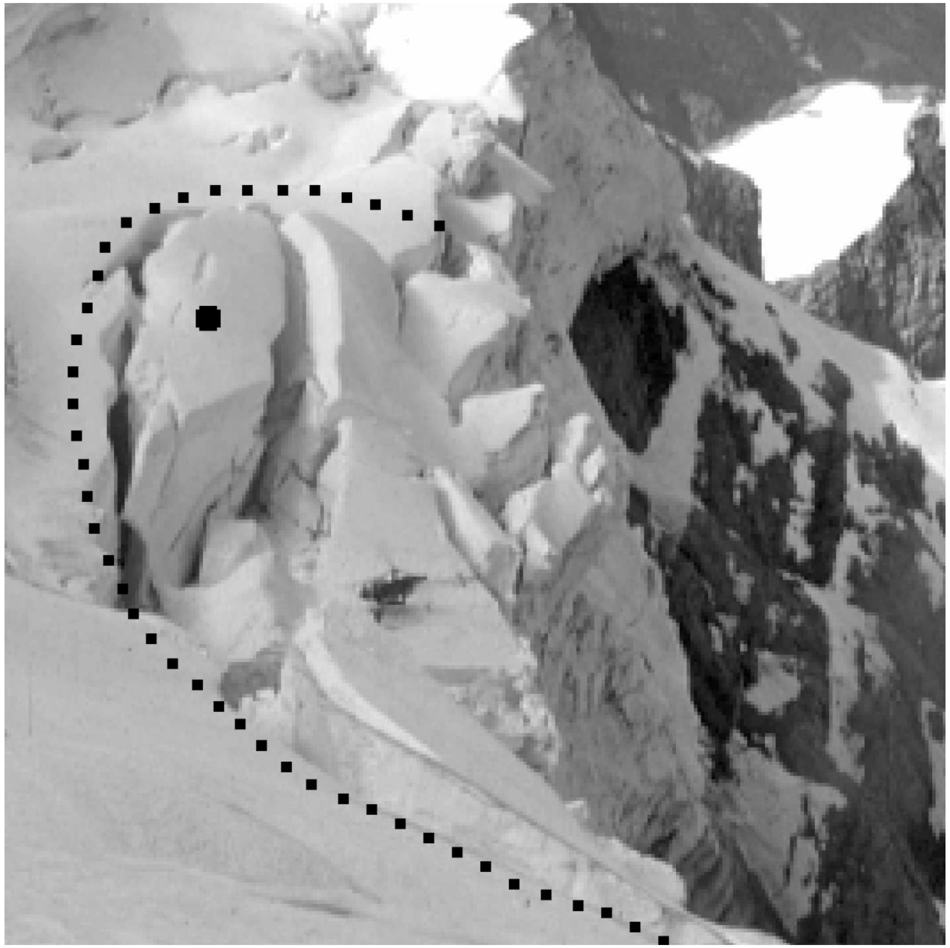
(Leonardo's maxim)

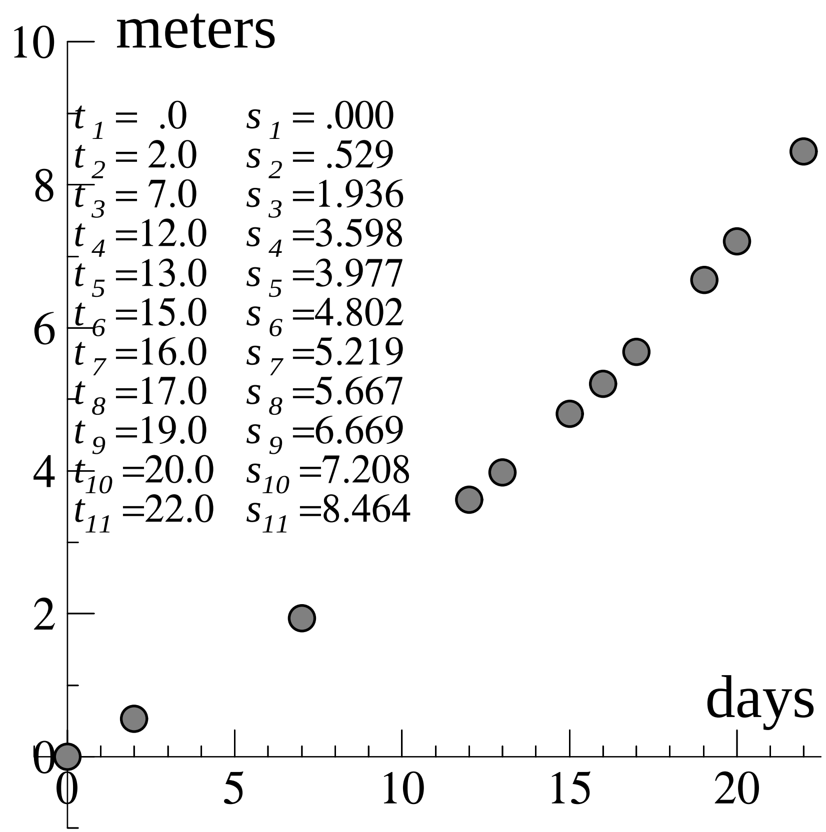
$$F = \sum_i (Ax_i^2 + 2Bx_iy_i + Cy_i^2 - Dx_i - Ey_i - 1)^2 = \text{min !}$$

The Baptism of Christ from 1472:



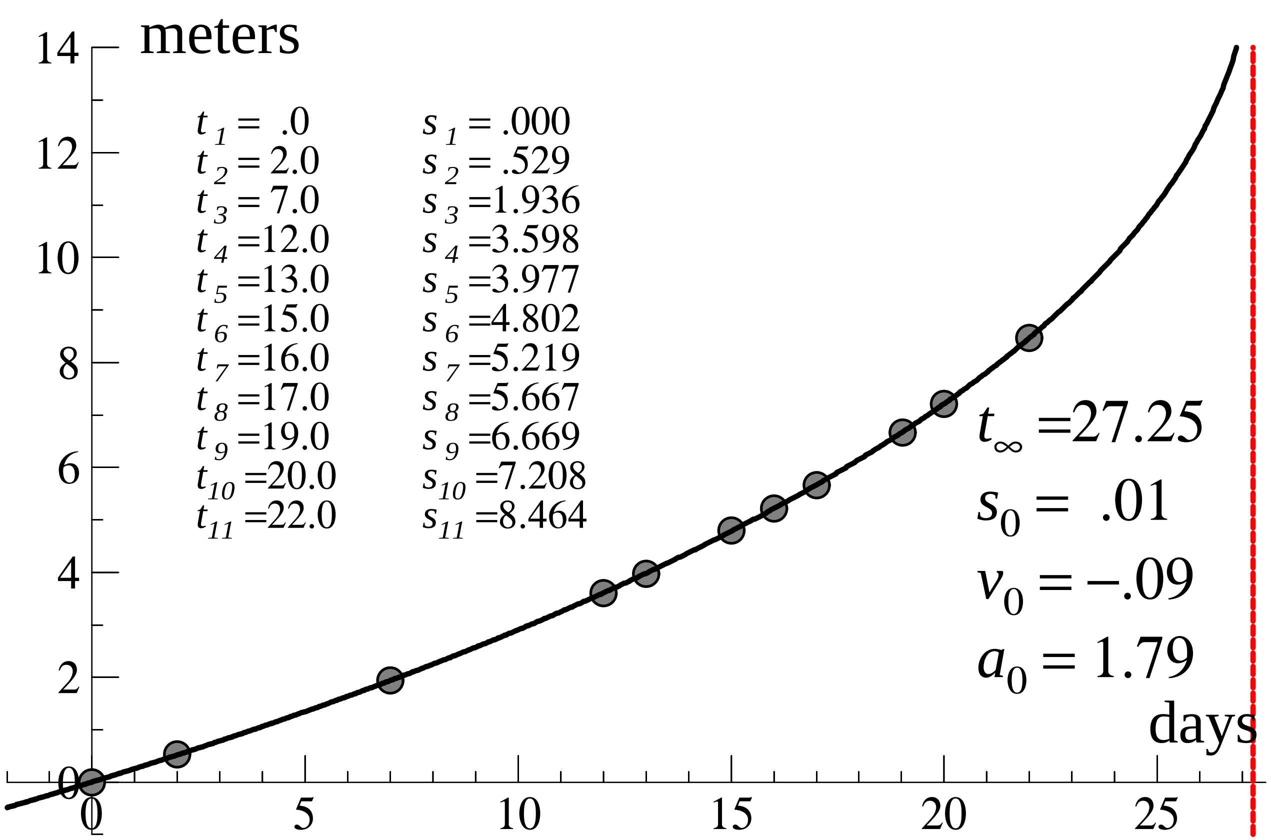
The Hanging Glacier above Grindelwald.





(Photo: M. Funk, ETHZ).

Problem: When will the glacier fall down?



$$v(t) = v_0 + \frac{a_0}{(t_\infty - t)^n}, \quad s(t) = s_0 + v_0 t + a_0 \left(\frac{(t_\infty - t)^{1-n} - t_\infty^{1-n}}{n-1} \right).$$

Predicted the ice fall for $t_\infty = 27.25 =$ 14th of August at 1 p.m.

Fell 14th of August at 2 a.m.