# How Euler Became Famous

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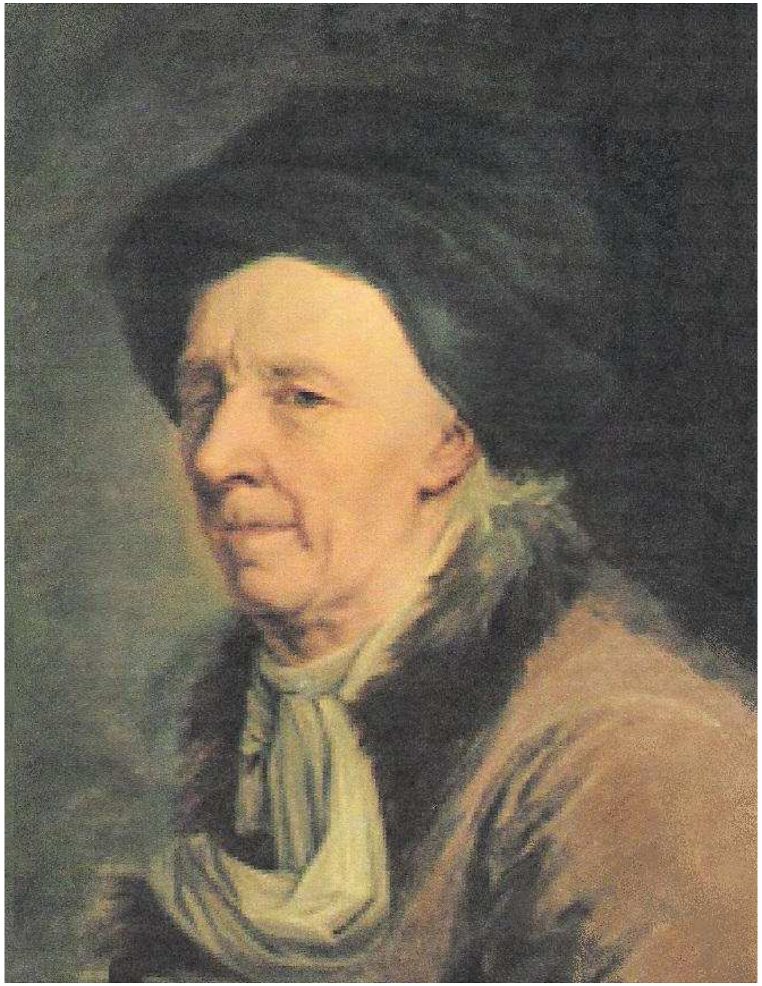
1707  1783

**Ph. Henry & G. Wanner**

**Bellinzona, Agosto 2007.**

## Euler's Work.

866 Publications E1 — E866 (Eneström).

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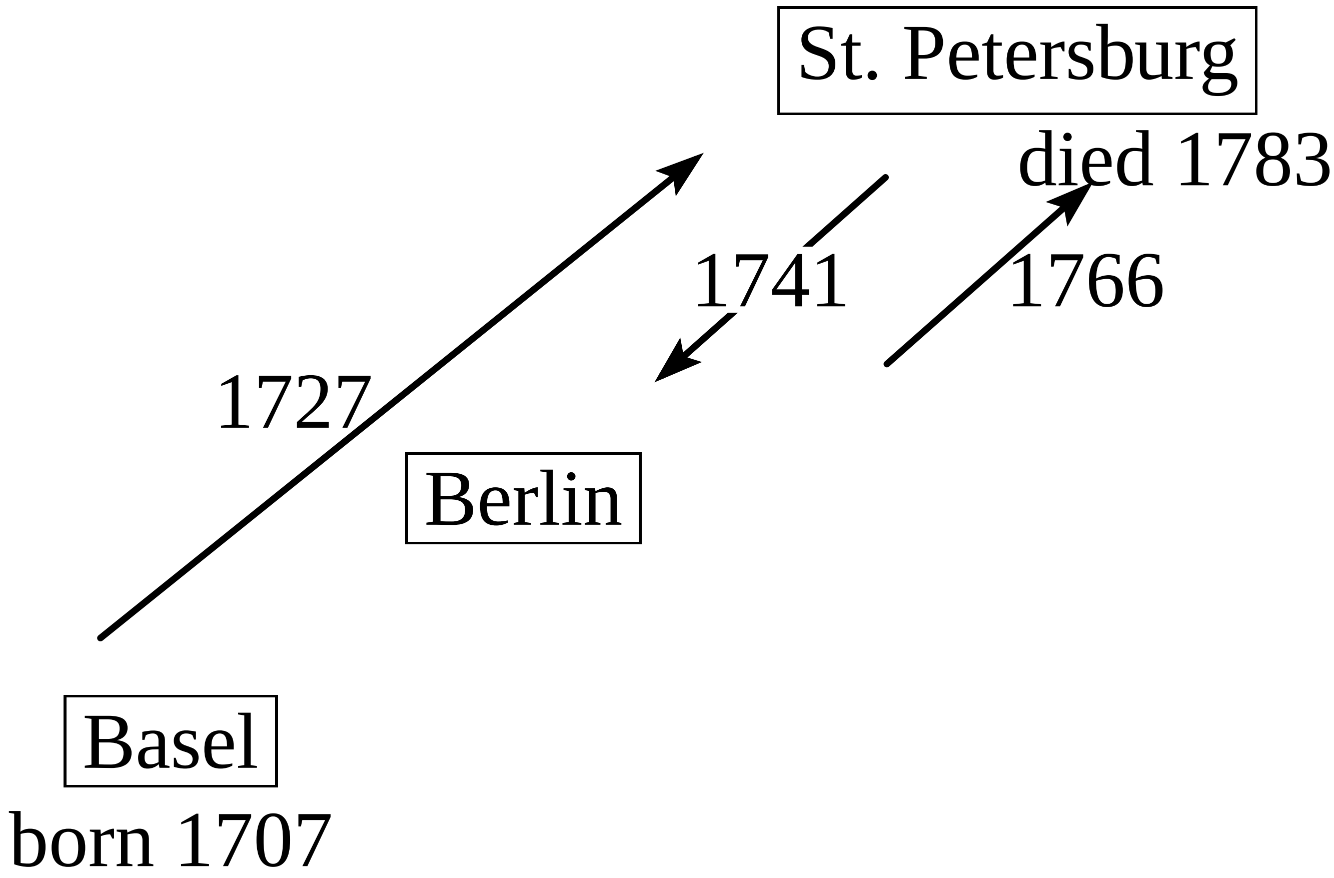
## Very Short Biography:



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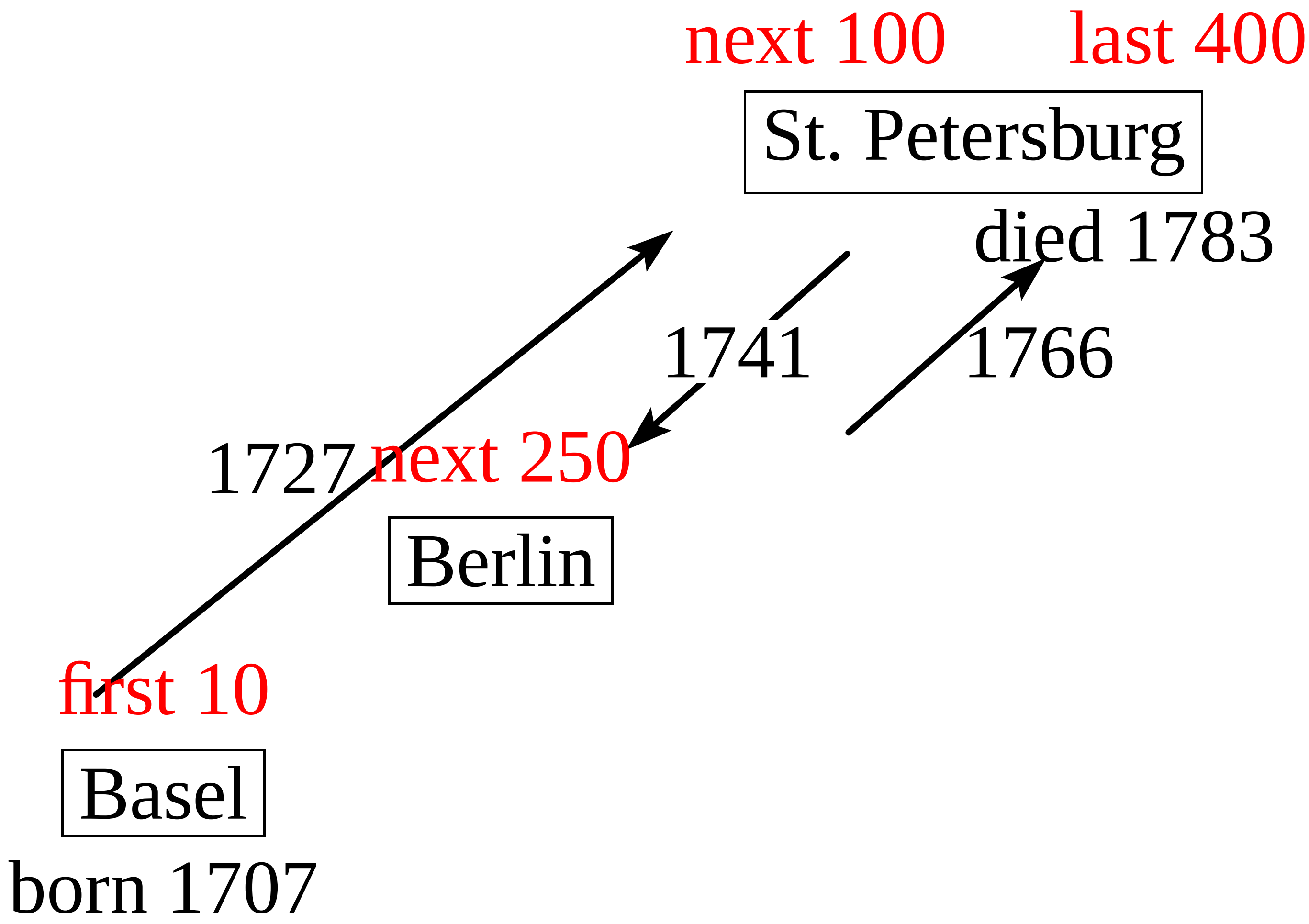


# Euler's Work.

866 Publications E1 — E866 (Eneström).

## List of results, formulas, theorems by Euler

(established together with A. Robert, Neuchâtel)

## Arithmetic

- Fermat number $2^{32}+1$ "divisibilis per 641" (E26 ; E134) ;
- $a^n + b^n = c^n$ impossible (Fermat $n = 3, 4$) (E98 ; E388) ;
- proofs of "small" Fermat (E134, E262) ;
- Divisor sums

$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \dots$ (E175) ;

- Theta Function

$(1-x)(1-x^2)(1-x^3)\dots = \sum_{n\in\mathbf{Z}}(-1)^n x^{\frac{n(3n-1)}{2}}$ (E244) ;

- Perfect numbers, Theorem of Euclid-Euler (E798) ;
- Amicable numbers (E100, E152) ;

- Numbers as sum of 2 squares (E228, E241) ;
- Numbers as sum of 4 squares (E242, E445) ;
- Euler Indicator $\varphi(n)$ (E271 ; E449) ;
- Theorem of Fermat-Euler: $n$ divides $a^{\varphi(n)} - 1$ (E271) ;
- Primitive roots for $p$ (E449) ;
- Proof of Wilson's Theorem (E560) ;
- Law of quadratic reciprocity, Euler criterion

$a^{\frac{p-1}{2}} = \left(\frac{a}{p}\right) \bmod p$ (E552) ;

- Euler's product of $\zeta$ Function : $\displaystyle\sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^s}$ ;
- divergence of $\sum_p \frac{1}{p}$ (E72);
- Generating function of Bernoulli numbers (E47, E212)

$$\frac{u}{e^u - 1} = 1 + \frac{B_1}{1!}u + \frac{B_2}{2!}u^2 + \frac{B_3}{3!}u^3 + \frac{B_4}{4!}u^4 + \dots ;$$

- Partitions of integers, corresp. generating function (E101)

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = \sum_{n\geq 0} p(n)x^n\,;$$

- Magic squares (E530) ;

## Topology

- Euler's polyhedra formula $f + s = a + 2$, Euler-Poincaré characteristic (E230) ;
- Königsberg bridges, graphs and Euler circuits (E53) ;
- Knight trajectory on Chess-board (Hamiltonian circuit) ;

## Geometry

- Euler-Bramagupta formula for circular quadrilateral (E135) ;
- Euler's identity for general quadrilateral $a^2 + b^2 + c^2 + d^2 = \ell_1^2 + \ell_2^2 + 4e^2$ (E135) ;
- Euler line in a triangle, relation of Chapple $d^2 = R^2 - 2rR$, Theorem and circle of Feuerbach (E325) ;

- Quadratic forms in $R^3$ to diagonal position (E102) ;
- Trigonometric functions; trigonometric identities ; product for $\sin nx$ (E246) ;
- New access to spherical trigonometry (E524) ;
- Number of intersections of two 3rd degree curves (Paradox of Euler-Cramer) ; approach of Bézout’s theorem (E147) ;
- Revolution Surfaces and their geodesics (E214) ;
- Euler-Meusnier formula for curvature of surface (E333)

$$\frac{1}{R} = \frac{\cos^2 \varphi}{R_1} + \frac{\sin^2 \varphi}{R_2} \, ;$$

- Euler angles of an orthogonal transformation of $R^3$ ;

**Analysis**

- Euler’s number $e = \left(1 + \frac{1}{\omega}\right)^{\omega} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots$ (E10, E15, E61, E101);

- Continued fractions $e-1$, $\frac{e+1}{e-1}$, irrationnality of $e$ (E 71, E101, E123) ;
- The famous relation (E61, E101)

$$e^{ix} = \cos x + i \sin x \,, \qquad e^{i\pi} = -1 \,;$$

- Logarithms of $-x$ and of complexes numbers ; $i^i$ (E168) ;
- Infinite product of $\sin z$, partial fraction decomposition of $\cot z$ (E41; E61);
- The “Basel problem” (E41, E63, E61, E464, E704)

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum \frac{1}{n^6} = \frac{\pi^6}{945}, \ldots \;;$$

- Euler-Maclaurin summation formula, Euler-Mascheroni constant $C$ (E47, E212, E642);
- Euler identities for derivatives of homogeneous fonctions (E212) ;
- Euler substitutions for integrals

$$\int R(\sin x, \cos x)\, dx, \quad \int R(\sqrt{ax^2+bx+c}, x)\, dx$$ (E352) ;

- Gamma function $\Gamma(x)$, Beta function $B(p,q)$ (E19, E254, E321, E421) ;
- Dilogarithm $\mathrm{Li}_2(x) = \sum \frac{z^n}{n^2}$ (E20, E342, E736) ;
- Transformation of double integrals (E391) ;
- Derivative of integrals w.r. to parameter (E464) ;
- Differential Equations with constant coefficients (E62);
- Inhomogeneous differential Equations with constant coefficients (E188)
- Differential Equations with weak singularities, hypergeometric series (E366, E710);
- Euler's method for general differential equations (E342) ;
- Taylor methods for general differential equations (E342) ;
- Euler-Fourier formula for Fourier series (E704) ;
- Variational calculus, Euler-Lagrange equations (66 detailed exemples) (E65) ;

## Physics

- Differential equations of mechanics (E10, E15, E177) ;
- Equilibrium of ships (*Scientia Navalis*) (E110, E111) ;
- Euler's equations for hydro- and aerodynamics (⇒ Navier-Stokes) (E225, 226, 227).
- Inertial ellipsoid of a rigid body (E291) ;
- New principle for angular momentum (E479) ;
- Euler equations for the rigid body ("corporum solidorum") (E292, E289),

. . . and 20 volumes of astronomical calculations.

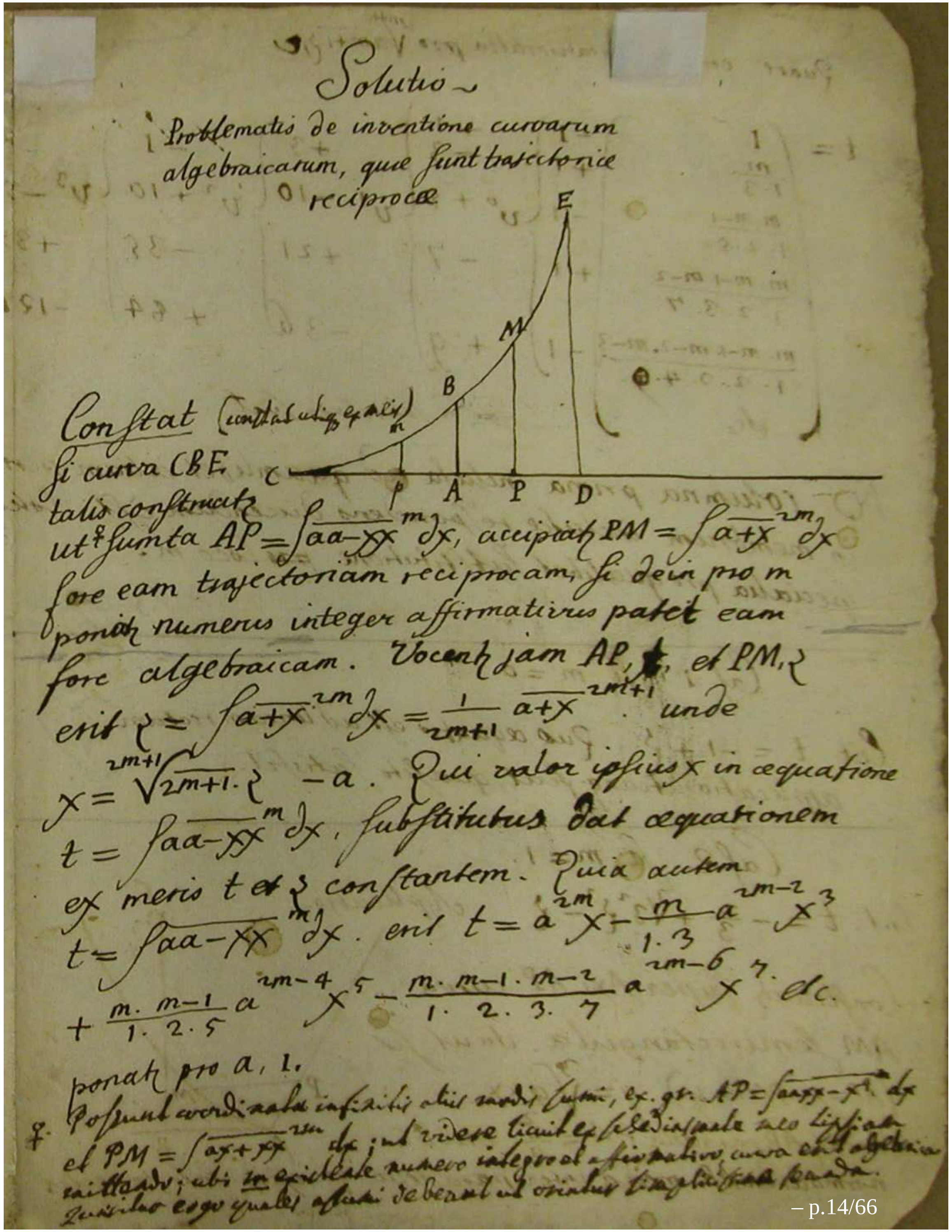
## His First Great Achievements.

- E3, E5 : *Trajectorias reciprocas . ...* (1725);
- E20, E41, E61 : $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \ldots = \frac{\pi^2}{6}$ (1735);
- E19, E254, E321, E421 : *The Gamma Function ...* (1729);
- E72 : *Prime Number Theorem ...* (1737);

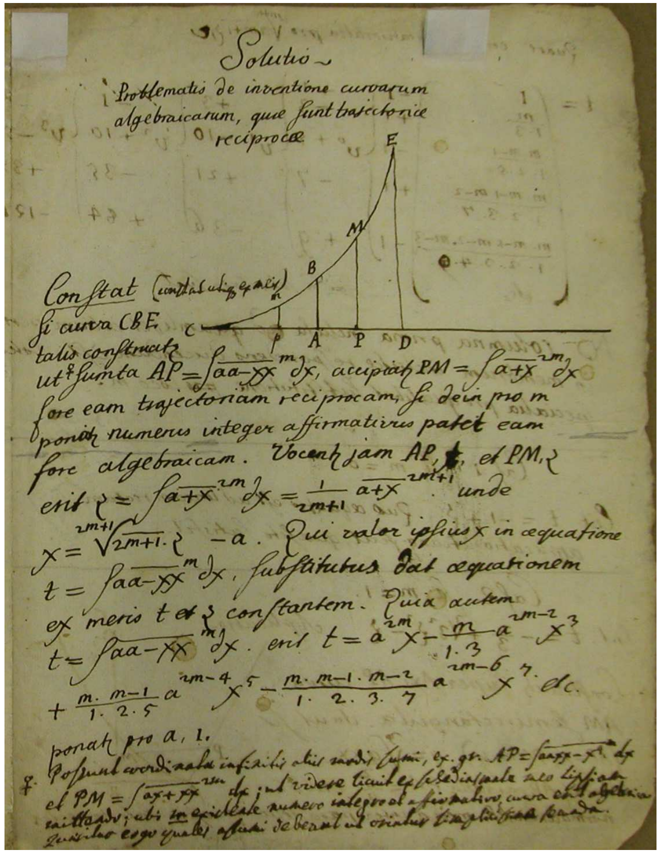
# Reciprocal Trajectories

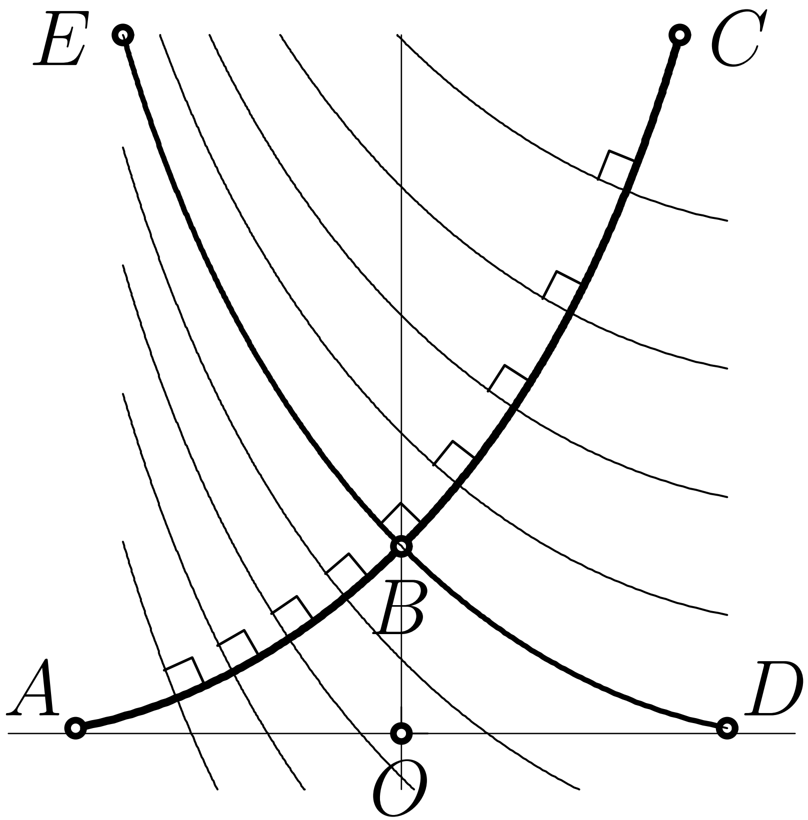
## (E3, E5)

Bodmer manuscript:
(found in Libr. in Geneva)



**Challenge.** (Leibniz, Nicolas II Bernoulli (1720):
invenire & construere curvam $ABC$, eandemque $DBE$, sed inverso situ positam ; (...) motu sibi semper parallelo, curvae $ABC$ & $DBE$ secent constanter se mutuo ad angulos rectos...




**New Challenge.** (Anonymous) Find simplest **algebraic** curve.

**Solution.** (by Joh. Bernoulli) Curve of third degree.

**New Challenge.** (Joh. Bernoulli) Find **second** simpl. alg. curve.

# 18 years old Euler challenges H. Pemberton in Cambridge:

"I must make reference to the publication by that Anonymo illi Angelo who posed the questions about reciprocal trajectories (...) I myself have devised a method for finding a general series of curves (...) All of which I will reveal in a year's time."

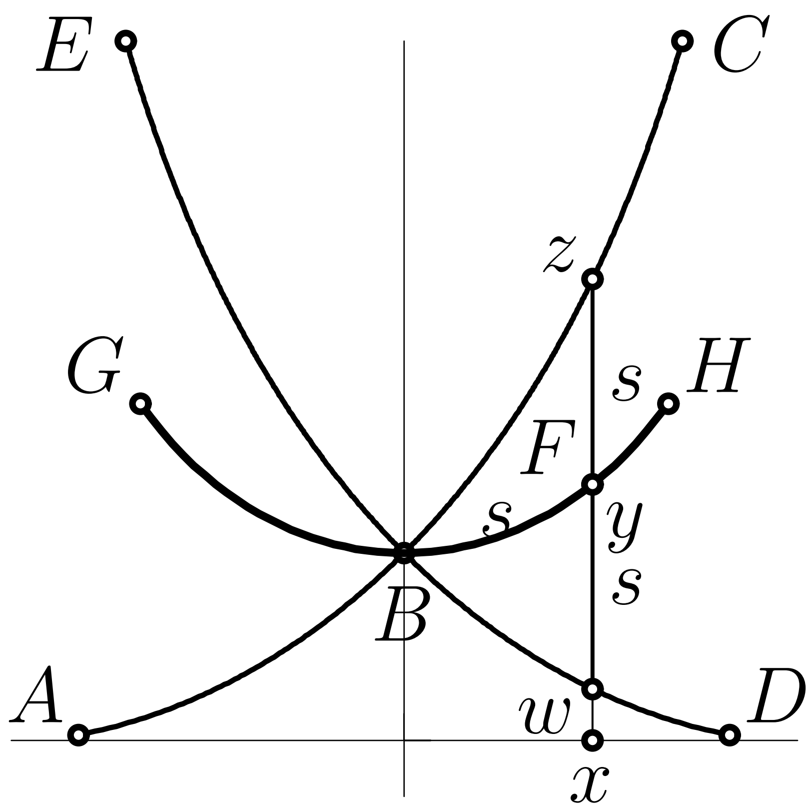
(Euler, end of E1, announcing E3. Transl. by Ian Bruce.)

$E3:$

$$z = y + s$$
$$w = y - s$$

$$dz = dy + ds$$
$$dw = dy - ds$$

$$\frac{dz}{dx} \cdot \frac{dw}{dx} = -1$$
$$\frac{dy^2}{dx^2} - \frac{ds^2}{dx^2} = -1$$
$$ds^2 = dx^2 + dy^2 \; .$$




$\Rightarrow$ Any symmetric curve $GBH$ leads to a solution.

# The curve of the ingenii Juvenis Leonhardus EULERUS (E3).

Without any explanation, Euler puts for the even function $y(x)$

$$x^2 + \frac{2}{3}\,a^2 = a\sqrt[3]{ay^2} \qquad \text{or} \qquad y = \frac{\left(x^2 + \frac{2}{3}a^2\right)^{\frac{3}{2}}}{a^2}\,,$$

$$dy = \frac{3\left(x^2 + \frac{2}{3}\,a^2\right)^{\frac{1}{2}} \cdot x\,dx}{a^2} \;\Rightarrow\; s = \int \frac{(3x^2 + a^2)\,dx}{a^2} = \frac{x^3}{a^2} + x\,.$$

$$z = y + s \;\Rightarrow\; (a^2 z - x^3 - a^2 x)^2 = \left(x^2 + \frac{2}{3}\,a^2\right)^3\,.$$

quae dividendo per $a^2$ ... seu ponendo $a^2 = \frac{3}{2}$ ad hanc

$$12x^3 z + 3x^2 + 18xz - 9z^2 + 4 = 0\,.$$

Quae est aequatio ad quarti ordinis curvam.

# Understand Bodmer man.:

$$x = \int (a^2 - t^2)^m \, dt$$

$$z = \int (a + t)^{2m} \, dt$$

$$w = -\int (a - t)^{2m} \, dt$$

orth. cond. $dz \cdot dw = -dx^2$

$$z = \frac{1}{2m+1}(a + t)^{2m+1}$$

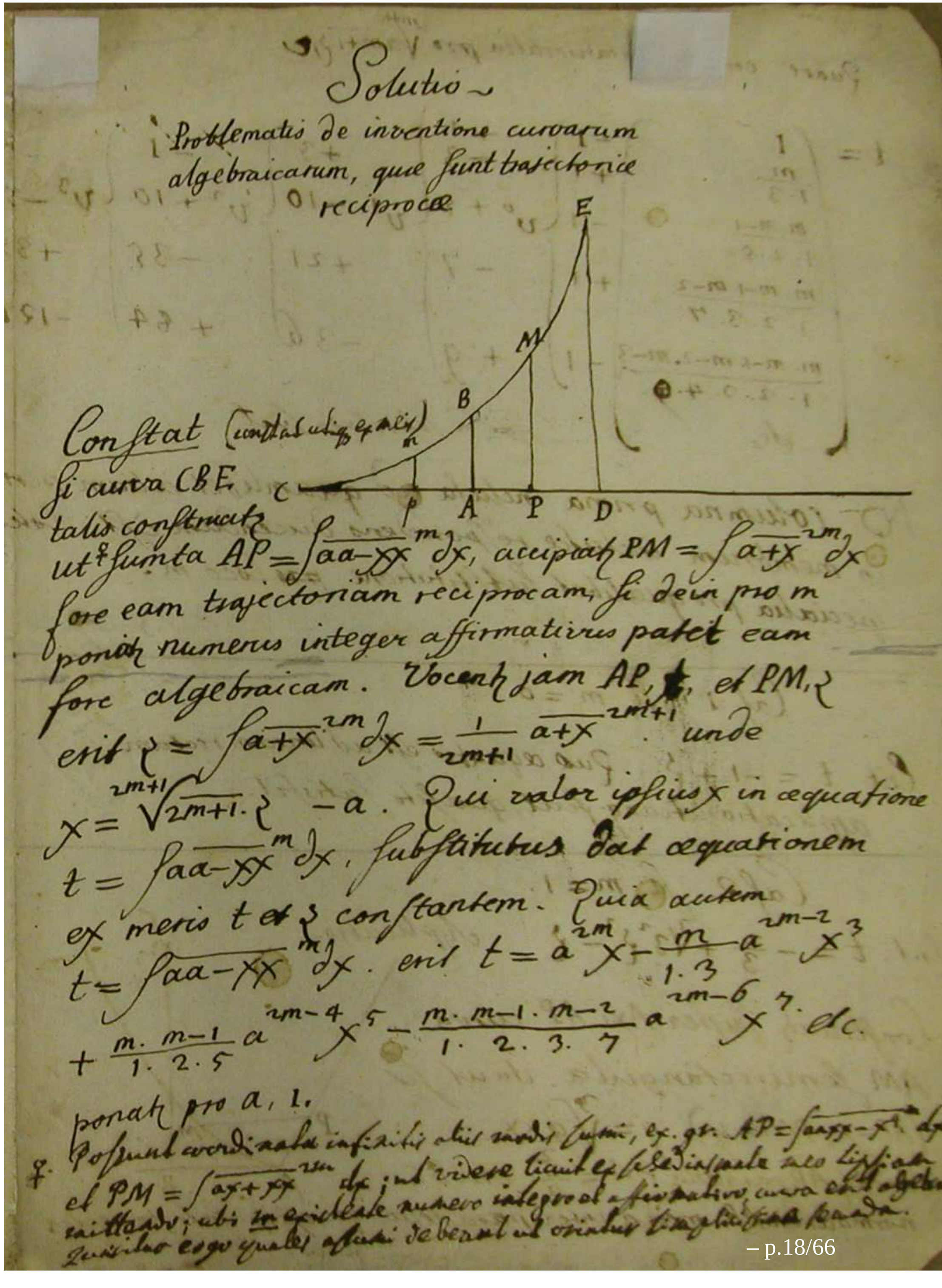
$$t = \sqrt[2m+1]{(2m+1)z} - a$$

and insert ($a = 1$ et $m = 1$) :

$$x = t - \tfrac{1}{3}t^3$$

$$x = -\tfrac{2}{3} + \sqrt[3]{9z^2} - z \, ,$$

(Sol. of Joh. Bernoulli)



Quare erit. brevitatis gratia pro $\sqrt[2m+1]{2m+1}\,\zeta$, $v$

$$t = \left(\begin{array}{l} 1 \\ \frac{m}{1\cdot 3} \\ \frac{m.m-1}{1\cdot 2\cdot 5} \\ \frac{m.m-1.m-2}{1\cdot 2\cdot 3\cdot 7} \\ \frac{m.m-1.m-2.m-3}{1\cdot 2\cdot 3\cdot 4\cdot 9} \\ \text{etc.} \end{array}\right) \begin{array}{r} -1 \\ +1 \\ -1 \\ +1 \\ -1 \end{array} \left\{\begin{array}{r} +1 \\ -3 \\ +5 \\ -7 \\ +9 \end{array}\right\} v^0 \left\{\begin{array}{r} +3 \\ -10 \\ +21 \\ -36 \end{array}\right\} v^1 \left\{\begin{array}{r} -1 \\ +10 \\ -35 \\ +84 \end{array}\right\} v^2 \left\{\begin{array}{r} -5 \\ +35 \\ -126 \end{array}\right\} v^3 \left\{\begin{array}{r} +1 \\ -21 \\ +126 \end{array}\right\} v^4 \left\{\begin{array}{r} +7 \\ -84 \end{array}\right\} v^5 \left\{\begin{array}{r} -1 \\ +36 \end{array}\right\} v^6 \; v^7 \; -9\,v^8 \; +1\,v^9$$

Hae in [illegible] sunt.

NB columna prima inclusa ad quamlibet aliam pertinet quodlibet scilicet membrum sequentium columnarum in membrum inclusae respondens ducendum est. Caeterum lex progressionis patet. Descendamus ad exempla specialia pro qualibet substitutione pro loco m.

---

Cas. 1. si $m = 0$

Erit $t = -1 + \zeta$. Quae aequatio est ad lineam rectam, applicatio utcunque fiat, quaesito satisfit.

---

Cas. 2. si $m = 1$.

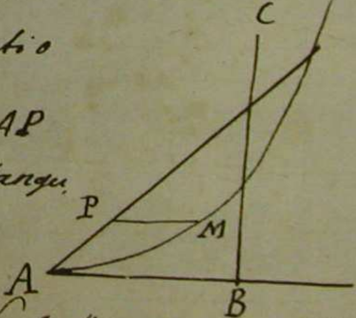
Erit. $t = -\frac{2}{3} + \sqrt[3]{3^2}\,\zeta^2 - \zeta$. Applicatio.


Construatur super Axi AP curva AM semirectangula. ita ut sit ($AP = x$. $PM = y$). $y = \sqrt[3]{\frac{9xx}{2}}$ ducatur AB parallela $PM = \frac{2}{3}$ erit BC normalis in AB. media inter parallelas quaesito satisfacientes

Cas. 3. si $m = 2$

Erit $t = -\frac{8}{15} + \frac{4}{3}\sqrt[5]{5^3}\,\zeta^3 - \sqrt[5]{5^4}\,\zeta^4 + \zeta$

Constructio




Construatur super Axi AB curva AM sesquirectangula ita ut sit ($AP = x$. $PM = y$)

$y = \frac{4}{3}\sqrt[5]{\frac{5^3 x^3}{2^3}} - \sqrt[5]{\frac{5^4 x^4}{2^4}}$ ducatur AB parall. PM $= \frac{8}{15}$ erit BC in AB normalis media inter parallelas quaesito satisfacientes.

Sed multum abest quominus haec curva sit in ordine simplicissimarum secunda, habeo etiam aliam multo simpliciorem quae ad [illegible] [illegible] [illegible] [illegible], sed [illegible] [illegible] [illegible].

## **The article E173.** Hundred's of solutions, for ex.

20 years later (viginti abhinc annis et quod excurrit), in *Nova methodus inveniendi traiectorias reciprocas algebraicas*, Euler computes hundreds of solutions, for example

$$9y^2 = 6y(2x^3+3x)+3x^2+4$$

$$64y^2 = 16y(8x^4+12x^2+3)-8x^2-9$$

$$225y^2 = 30y(24x^5+40x^3+15x)+15x^2+16$$

$$576y^2 = 48y(64x^6+120x^4+60x^2+5)-24x^2-25$$

$$1225y^2 = 70y(160x^7+336x^5+210x^3+35x)+35x^2+36$$

$$2304y^2 = 96y(384x^8+896x^6+672x^4+168x^2+7)-48x^2-49$$

$$3969y^2 = 126y(896x^9+2304x^7+2016x^5+672x^3+63x)+63x^2+64$$

quae ergo aequationum series, quousque libuerit, facile continuabitur.

**Next Challenge:** Find value of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = \ ?$$

(“Basel” Problem; P. Mengoli (1650), Jac. Bernoulli (1689))

Jakob Bernoulli
(1654-1705)

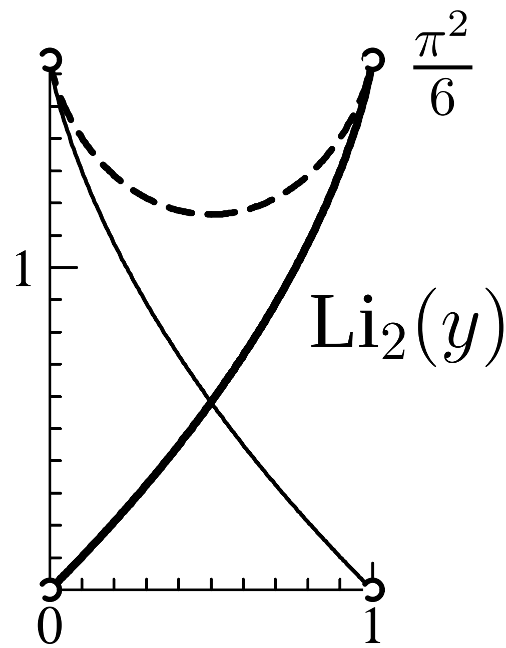
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \text{etc.} = ?$$

“La recherche approfondie de la somme [de cette série] est plus difficile que quelqu'un s'y attendrait, et pourtant nous recueillons [d'après ce qui précède] qu'elle est finie (...) : si quelqu'un trouve, qu'il nous communique ce qui jusqu'ici a échappé à notre industrie, qu'il accepte de nous de grandes grâces.”
(Jakob Bernoulli, *Tractatus de Seriebus Infinitis)*

“Utinam Frater superstes esset !”
(Johann Bernoulli)

## E20: At least a good num. value by inventing the Dilogarithm

$$\mathrm{Li}_2(y) = \int -\frac{\ell(1-y)}{y}\,dy = \frac{y}{1} + \frac{y^2}{4} + \frac{y^3}{9} + \ldots$$




(the "Radiation Integral")

$$\mathrm{Li}_2(1) = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

It is enough to compute

$$\mathrm{Li}_2(\tfrac{1}{2}) = \frac{1}{1\cdot 2} + \frac{1}{4\cdot 4} + \frac{1}{9\cdot 8} + \frac{1}{16\cdot 16} + \ldots$$

because

$$\mathrm{Li}_2(y) + \mathrm{Li}_2(1-y) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots - \ell(y)\cdot\ell(1-y)\,.$$

"ergo somma serie est $= 1.644934$ quam proxime."

## The break-through E41: (1735)

"One of Euler's most sensational early discoveries, perhaps the one which established his growing reputation most firmly, was his summation of the series $\sum_1^\infty n^{-2}$ (...). This was a famous problem, first formulated by P. Mengoli in 1650 ; it had resisted the efforts of all earlier analysts, including Leibniz and the Bernoullis." (A. Weil, *Number theory*, 1984, p. 184)

First idea: The "polynomial" (in $1/s^2$)

$$\frac{\sin s}{s} = 1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots$$

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**Back to first year algebra:** The polynomial

$$p = x^3 - \alpha x^2 + \beta x - \gamma$$

$\Rightarrow \quad \alpha = x_1 + x_2 + x_3$ (Viète)

$\Rightarrow \quad p = (x - x_1)(x - x_2)(x - x_3)$ (Descartes)

$\Rightarrow \quad x_1^2 + x_2^2 + x_3^2 = \alpha^2 - 2\beta$ (Newton)

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$$\frac{\sin s}{s} = 1 - \frac{s^2}{1\cdot 2\cdot 3} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \ldots$$

has "roots" $\frac{1}{\pi^2}, \frac{1}{4\pi^2}, \frac{1}{9\pi^2}, \ldots$. Hence, by Viète,

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots = \frac{1}{6} \quad \text{or} \quad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \ldots = \frac{\pi^2}{6}.$$

and by Descartes

$$\sin s = s(1 - \frac{s^2}{\pi^2})(1 - \frac{s^2}{4\pi^2})(1 - \frac{s^2}{9\pi^2})(1 - \frac{s^2}{16\pi^2}) \ldots .$$

Furthermore, the theoremata NEUTONIANI give

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{90}$$

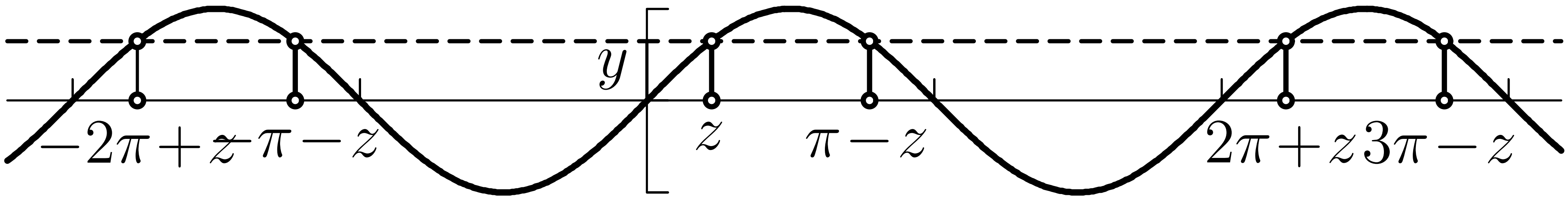
$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \ldots = \frac{\pi^6}{945}$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \ldots = \frac{\pi^8}{9450}$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \ldots = \frac{\pi^{10}}{93555}$$

Second idea. Choose $y = \sin z$, then the "polynomial" (in $1/s$)

$$0 = 1 - \frac{s}{y} + \frac{s^3}{1 \cdot 2 \cdot 3 \cdot y} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot y} + \ldots$$




has "roots"

$$\frac{1}{z}, \; \frac{1}{\pi - z}, \; \frac{1}{-\pi - z}, \; \frac{1}{2\pi + z}, \; \frac{1}{-2\pi + z}, \ldots$$

so by Viète,

$$\frac{1}{\sin z} = \ldots + \frac{1}{z + 2\pi} - \frac{1}{z + \pi} + \frac{1}{z} - \frac{1}{z - \pi} + \frac{1}{z - 2\pi} - \ldots$$

and inserting e.g., $z = \frac{\pi}{2}$ and $y = 1$, we have

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}$$

and the theoremata NEUTONIANI , give

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \ldots = \frac{\pi^3}{32}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \ldots = \frac{5\pi^5}{1536}$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \ldots = \frac{61\pi^7}{184320}$$

and so on.

The proofs were criticized by Joh. Bernoulli, Daniel Bernoulli and Gabriel Cramer. So, Euler searched for more rigorous proofs .

**E59:**

$$\int_0^1 \frac{x^{m-1}+x^{n-m-1}}{1+x^n}\,dx = \frac{\pi}{n}\,\frac{1}{\sin\frac{m\pi}{n}}$$
$$= \ldots + \frac{1}{m+2n} - \frac{1}{m+n} + \frac{1}{m} - \frac{1}{m-n} + \frac{1}{m-2n} - \ldots$$

**E162:** General method for integration of rational functions $\displaystyle\int \frac{P(x)\,dx}{Q(x)}$ by partial fraction decomposition and direct verification of this integral.

**E60:** Need roots of unity and factorizations of the type

$$z^n - a^n = (z-a)\prod_{k=1}^{(n-1)/2}\left(z^2 - 2az\cos\frac{2k\pi}{n} + a^2\right).$$

**E61:** The formula

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{1}{2i}((1 + \frac{ix}{n})^n - (1 - \frac{ix}{n})^n)$$

by inserting the above factorizations of $z^n - a^n$ one obtains

$$\sin x = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2})(1 - \frac{x^2}{16\pi^2})\ldots$$

and similarly

$$\cos x = (1 - \frac{4x^2}{\pi^2})(1 - \frac{4x^2}{9\pi^2})(1 - \frac{4x^2}{25\pi^2})(1 - \frac{4x^2}{49\pi^2})\ldots .$$

**E130 :** Take logarithms of these products, and differentiate
$\Rightarrow$ partial fraction decomp. of $\frac{1}{\sin x}$ and $\cot x$.

# E47, E212: Connection with the Euler-Maclaurin Formula

$$\sum_{i=1}^{n} f(i) = \int_0^n f(x)\,dx + \frac{1}{2}(f(n) - f(0)) \quad + \\ + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(0)\right).$$

Using a generating function (invented here) Euler gets

$$z \cdot \cot z = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2\pi^2} = \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{B_{2\ell}}{(2\ell)!} 2^{2\ell} z^{2\ell}$$

and by expanding the geometric series we see finally

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\ell}} = (-1)^{\ell-1} \frac{(2\pi)^{2\ell}}{2 \cdot (2\ell)!} B_{2\ell}\ .$$

“So hat Euler nicht nur zuerst die Summen der reziproken geraden Potenzen der natürlichen Zahlen bestimmt, sondern auch den Zusammenhang der dabei auftretenden Koeffizienten mit anderen wichtigen Formeln der Analysis nachgewiesen. Seine Untersuchungen über diesen Gegenstand gehören zu den schönsten und tiefsten, mit denen uns sein Genius beschenkt hat”

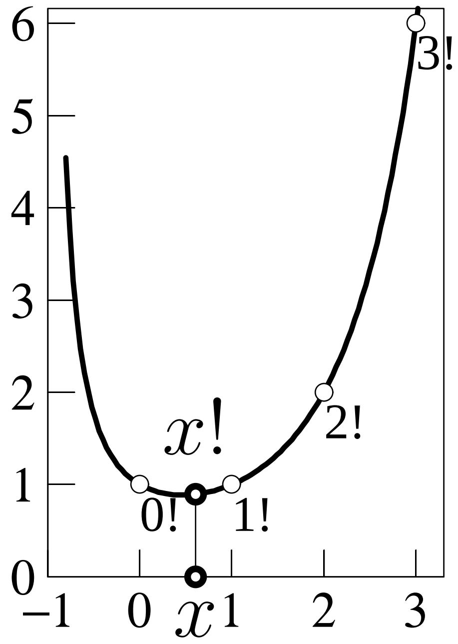
(P. Stäckel, Bibl. Math., 8 (1907-1908)

## Third Challenge: the Gamma Function E19.

Long discussions in correspondence between Chr. Goldbach and Dan. Bernoulli.

Problem. Find, for $x$, the value $x!$ "interpolating"

$$1\,, \quad 1\cdot 2\,, \quad 1\cdot 2\cdot 3\,, \quad 1\cdot 2\cdot 3\cdot 4\,, \quad \ldots$$



# Third Challenge: the Gamma Function E19.

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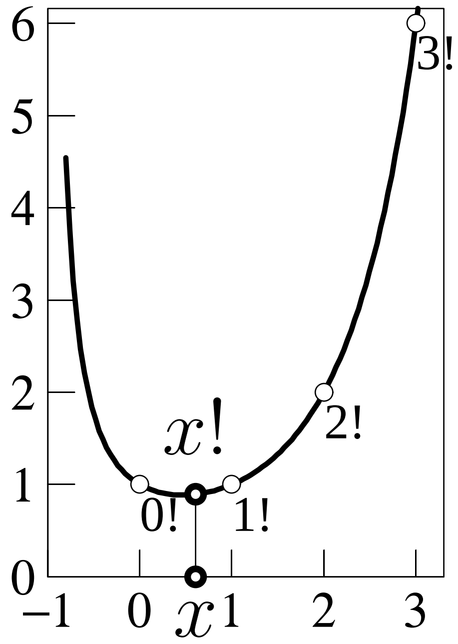


Problem. Find, for $x$, the value $x!$ "interpolating"

$$1\,,\quad 1\cdot 2\,,\quad 1\cdot 2\cdot 3\,,\quad 1\cdot 2\cdot 3\cdot 4\,,\quad \ldots$$

22 years old Euler states 1729 the solution:

$$x! = \frac{1\cdot 2^x}{1+x}\cdot\frac{2^{1-x}\cdot 3^x}{2+x}\cdot\frac{3^{1-x}\cdot 4^x}{3+x}\cdot\frac{4^{1-x}\cdot 5^x}{4+x}\cdots\,.$$

Special case $x=\frac{1}{2}$ (using Wallis' product):

$$\left(\frac{1}{2}\right)! = \sqrt{\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdots} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}\,.$$

How did Euler find this formula ?? Explanation in (E652):

Euler took inspiration from Wallis' Arithm. Infinitorum (1655)

$$I_n = \int_0^1 (1-x^2)^n \, dx \qquad \Rightarrow \qquad I_n = \frac{2n}{2n+1} \cdot I_{n-1}$$

| $I_0$ | $I_1$ | $I_2$ | $I_3$ | $\ldots$ |
|---|---|---|---|---|
| $1$ | $\frac{2}{3}$ | $\frac{2}{3} \cdot \frac{4}{5}$ | $\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7}$ | $\ldots$ |

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| | | | | | |
|---|---|---|---|---|---|
| $I_0$ | $I_{\frac{1}{2}}$ | $I_1$ | $I_2$ | $I_3$ | $\ldots$ |
| $1$ | $\alpha$ | $\frac{2}{3}$ | $\frac{2}{3} \cdot \frac{4}{5}$ | $\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7}$ | $\ldots$ |

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| $I_0$ | $I_{\frac{1}{2}}$ | $I_1$ | $I_{\frac{3}{2}}$ | $I_2$ | $I_{\frac{5}{2}}$ | $I_3$ | $\dots$ |
|---|---|---|---|---|---|---|---|
| $1$ | $\alpha$ | $\frac{2}{3}$ | $\frac{3}{4}\alpha$ | $\frac{2}{3}\cdot\frac{4}{5}$ | $\frac{3}{4}\cdot\frac{5}{6}\alpha$ | $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}$ | $\dots$ |

Quo facilius res ſuccedat, progreſſionis (ibidem repertæ) termini

$$\tfrac{1}{2}\square .\ 1 .\ \square .\ \tfrac{3}{2} .\ \tfrac{4}{3}\square .\ \frac{3\times 5}{2\times 4} .\ \frac{4\times 6}{3\times 5}\square .\ \frac{3\times 5\times 7}{2\times 4\times 6} .\ \text{\&c.}$$

$$\square \begin{cases} \text{minor quam } \dfrac{3\times3\times5\times5\times7\times7\times9\times9\times11\times11\times13\times13}{2\times4\times4\times6\times6\times8\times8\times10\times10\times12\times12\times14} \times \sqrt{1\tfrac{1}{13}} . \\ \text{major quam } \dfrac{3\times3\times5\times5\times7\times7\times9\times9\times11\times11\times13\times13}{2\times4\times4\times6\times6\times8\times8\times10\times10\times12\times12\times14} \times \sqrt{1\tfrac{1}{14}} . \end{cases}$$

**Idea.** Put $x! = \alpha$ and intercalate the sequence $x!,\ (1+x)!, \ldots$ :

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $0!$ | | $1!$ | | $2!$ | | $3!$ | | | $\ldots$ |
| | $x!$ | | $(1+x)!$ | | $(2+x)!$ | | $(3+x)!$ | | |
| $1$ | | $1$ | | $1 \cdot 2$ | | $1 \cdot 2 \cdot 3$ | | | $\ldots$ |
| | $\alpha$ | | $\alpha(1+x)$ | | $\alpha(1+x)(2+x)$ | | $\alpha(1+x)(2+x)(3+x)$ | | |

Idea. Put $x! = \alpha$ and intercalate the sequence $x!, \ (1+x)!, \ldots$ :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $0!$ | | $1!$ | | $2!$ | | $3!$ | | $\ldots$ |
| | $x!$ | | $(1{+}x)!$ | | $(2{+}x)!$ | | $(3{+}x)!$ | |
| $1$ | | $1$ | | $1\cdot 2$ | | $1\cdot 2\cdot 3$ | | $\ldots$ |
| | $\alpha$ | | $\alpha(1{+}x)$ | | $\alpha(1{+}x)(2{+}x)$ | | $\alpha(1{+}x)(2{+}x)(3{+}x)$ | |

“in infinitum continuata tandem cum geometrica confundatur”

$$(N{+}1)! = N!(N{+}1), \ \ (N{+}2)! = N!(N{+}1)(N{+}2) \approx N!{\cdot}(N{+}1)^2$$

hence $(N+x)! \approx N! \cdot (N{+}1)^x$

for example, if $N = 3$, inserting $(3+x)!$ we obtain

$$\alpha \approx \frac{1\cdot 2\cdot 3\cdot 4^x}{(1{+}x)(2{+}x)(3{+}x)} = \frac{1\cdot 2\cdot 3^x}{(1{+}x)(2{+}x)} \cdot \frac{3^{1-x}\cdot 4^x}{3{+}x} = \ldots$$

Integral Formulas. Because of presence of $\sqrt{\pi}$ no hope for algebraic formulas... Perhaps integral formulas may help!

Discovery:

$$\int_0^1 x^{m-1}(1-x)^n\,dx = \frac{n}{m}\cdot\frac{n-1}{m+1}\cdots\frac{1}{m+n-1}\cdot\frac{1}{m+n}$$

Substitution $x^m = y$: $\Rightarrow$

$$\int_0^1 (1-y^{\frac{1}{m}})^n\,dy = \frac{1}{m+1}\cdot\frac{2}{m+2}\cdots\frac{n}{m+n}$$

Divide both sides by $1/m^n$: $\Rightarrow$

$$\lim_{m\to\infty}\int_0^1 \left(\frac{1-y^{\frac{1}{m}}}{\frac{1}{m}}\right)^n dy = \int_0^1 (-\ln y)^n\,dy = n!$$

or (E675)

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\,dt \qquad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,.$$

# Products for Beta Function Integrals (E254)

$$I_k = \int_0^1 x^{m-1}(1-x^n)^{k-1}\,dx \;\Rightarrow\; I_{k+1} = \frac{kn}{m+kn}\cdot I_k,\;\; I_1 = \frac{1}{m}\,.$$

| $I_1$ | $I_2$ | $I_3$ |
|---|---|---|
| $\frac{1}{m}$ | $\frac{1}{m}\cdot\frac{n}{m+n}$ | $\frac{1}{m}\cdot\frac{n}{m+n}\cdot\frac{2n}{m+2n}$ |

# Products for Beta Function Integrals (E254)

$$I_k = \int_0^1 x^{m-1}(1-x^n)^{k-1}\,dx \quad \Rightarrow \quad I_{k+1} = \frac{kn}{m+kn}\cdot I_k,\;\; I_1 = \frac{1}{m}\,.$$

| $I_1$ | $I_k$ | $I_2$ | $I_3$ |
| --- | --- | --- | --- |
| $\frac{1}{m}$ | $\alpha$ | $\frac{1}{m}\cdot\frac{n}{m+n}$ | $\frac{1}{m}\cdot\frac{n}{m+n}\cdot\frac{2n}{m+2n}$ |

# Products for Beta Function Integrals (E254)

$$I_k = \int_0^1 x^{m-1}(1-x^n)^{k-1}\,dx \;\Rightarrow\; I_{k+1} = \frac{kn}{m+kn}\cdot I_k,\;\; I_1 = \frac{1}{m}.$$

$$\begin{array}{cccccccc} I_1 & I_k & I_2 & I_{k+1} & I_3 & I_{k+2} \\ \frac{1}{m} & \alpha & \frac{1}{m}\cdot\frac{n}{m+n} & \alpha\frac{kn}{m+kn} & \frac{1}{m}\cdot\frac{n}{m+n}\cdot\frac{2n}{m+2n} & \alpha\cdots \end{array}$$

"Same procedure as every year" ... $\Rightarrow$

$$\int_0^1 x^{m-1}(1-x^n)^{k-1}\,dx =$$

$$\frac{1}{m}\cdot\frac{1(m+kn)}{k(m+n)}\cdot\frac{2(m+kn+n)}{(k+1)(m+2n)}\cdot\frac{3(m+kn+2n)}{(k+2)(m+3n)}\cdots$$

## Symmetric Beta Function Products (E321)

set $m = p$ and $k = \frac{q}{n}$, then

$$\int_0^1 x^{p-1}\left(1 - x^n\right)^{\frac{q}{n}-1} dx =$$

$$\frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \cdot \frac{2n(p+q+2n)}{(p+2n)(q+2n)} \cdot \frac{3n(p+q+3n)}{(p+3n)(q+3n)} \cdots$$

## Relation of Beta Function Products with the Sinus (E321)

Set $p + q = n$ (i.e. $q = n - p$), then

$$\int_0^1 \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^p}} = \int_0^1 \frac{x^{q-1}\,dx}{\sqrt[n]{(1-x^n)^q}}$$

$$= \frac{1}{p} \cdot \frac{nn}{nn-pp} \cdot \frac{4nn}{4nn-pp} \cdot \frac{9nn}{9nn-pp} \cdots = \frac{\pi}{n \sin\frac{p\pi}{n}} = \frac{\pi}{n \sin\frac{q\pi}{n}} .$$

## Relation of the Gamma Function with the Sinus (E421)

After long calculations, Euler finally discovers that

$$(\lambda)! \cdot (-\lambda)! = \frac{\lambda\pi}{\sin \lambda\pi} \qquad \text{or} \qquad \Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin \lambda\pi}$$

and readily remarks an easy access by multipling the very first product of (E19):

$$(\lambda)! = \frac{1 \cdot 2^{\lambda}}{1+\lambda} \cdot \frac{2^{1-\lambda} \cdot 3^{\lambda}}{2+\lambda} \cdots \qquad (-\lambda)! = \frac{1 \cdot 2^{-\lambda}}{1-\lambda} \cdot \frac{2^{1+\lambda} \cdot 3^{-\lambda}}{2-\lambda} \cdots$$

factor by factor to obtain

$$\frac{1 \cdot 1}{1-\lambda^2} \cdot \frac{2 \cdot 2}{4-\lambda^2} \cdot \frac{3 \cdot 3}{9-\lambda^2} \cdots = \frac{\lambda\pi}{\sin \lambda\pi} \,.$$

# NUMBER THEORY.

"(...) feu Mr. Fermat a proposé plusieurs théorèmes sur la nature des nombres (...). Je serois fort curieux de voir [les démonstrations], car je suis obligé d'avouer, qu'aïant travaillé dans ces matieres plus de 14 ans, je n'ai pû trouver les démonstraions de tous. Ce seroit un grand avantage pour ceux qui aiment ces spéculations, et même pour la vérité,..."

(Lettre d'Euler à Clairaut, Berlin avril 1742)

"Je n'ai jamais entendu parler des Theoremes de Fermat ni de ce que peuvent être devenus ses papiers. Cette Matiere doit être fort epineuse ...."

(Lettre de Clairaut à Euler, Paris 29 mai 1742)

"Mais ne trouvez vous pas que c'est presque faire trop d'honneur aux nombres premiers que d'y répandre tant de richesses, et ne doit-on aucun égard au goût raffiné de notre siècle ?"

(Lettre de Dan. Bernoulli à Nic. Fuss, Bâle 18 mars 1778)

“One must realize that Euler had absolutely nothing to start from except Fermat’s mysterious-looking statements.”

(A. Weil, 1972)

“Die Grundlage zu allen Untersuchungen, welche den allgemeinen Teil der Zahlentheorie ausmachen, ist von *Euler* geschaffen.”

(P.L. Chebyshev 1889)

“If Euler had never done anything *except* number theory, he would still be remembered as one of the great mathematicians.”

(P. Erdös 1983)

# Prime Numbers.

"Les mathématiciens ont tâché jusqu'ici en vain à découvrir un ordre quelconque dans la progression des nombres premiers, et on a lieu de croire, que c'est un mystère auquel l'esprit humain ne saurait jamais pénétrer. Pour s'en convaincre, on n'a qu'à jeter les yeux sur les tables des nombres premiers,..."

(Euler E175, orig. in French)

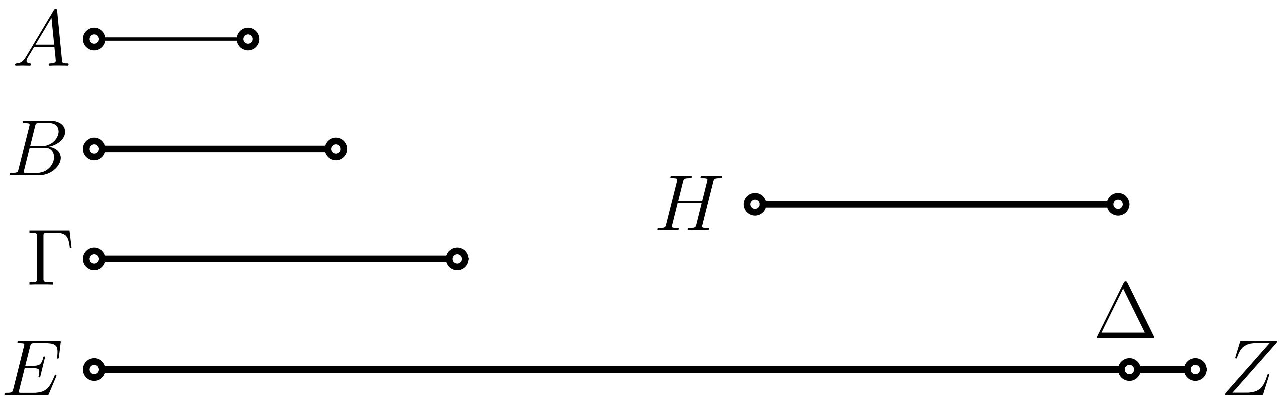
Definition. (Euclid, *Elements*, Book VII, Definition 11)

*Πρώτος αριθμός εστιν ο μονάδι μόνη μετρούμενος*

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 9 | 17 | 25 | 33 | 41 | 49 | 57 | 65 | 73 | 81 | 89 | 97 | 105 | 113 | 121 | 129 | 137 | 145 | 153 | 161 | 169 | 177 | 185 | 193 |
| 2 | 10 | 18 | 26 | 34 | 42 | 50 | 58 | 66 | 74 | 82 | 90 | 98 | 106 | 114 | 122 | 130 | 138 | 146 | 154 | 162 | 170 | 178 | 186 | 194 |
| 3 | 11 | 19 | 27 | 35 | 43 | 51 | 59 | 67 | 75 | 83 | 91 | 99 | 107 | 115 | 123 | 131 | 139 | 147 | 155 | 163 | 171 | 179 | 187 | 195 |
| 4 | 12 | 20 | 28 | 36 | 44 | 52 | 60 | 68 | 76 | 84 | 92 | 100 | 108 | 116 | 124 | 132 | 140 | 148 | 156 | 164 | 172 | 180 | 188 | 196 |
| 5 | 13 | 21 | 29 | 37 | 45 | 53 | 61 | 69 | 77 | 85 | 93 | 101 | 109 | 117 | 125 | 133 | 141 | 149 | 157 | 165 | 173 | 181 | 189 | 197 |
| 6 | 14 | 22 | 30 | 38 | 46 | 54 | 62 | 70 | 78 | 86 | 94 | 102 | 110 | 118 | 126 | 134 | 142 | 150 | 158 | 166 | 174 | 182 | 190 | 198 |
| 7 | 15 | 23 | 31 | 39 | 47 | 55 | 63 | 71 | 79 | 87 | 95 | 103 | 111 | 119 | 127 | 135 | 143 | 151 | 159 | 167 | 175 | 183 | 191 | 199 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 | 88 | 96 | 104 | 112 | 120 | 128 | 136 | 144 | 152 | 160 | 168 | 176 | 184 | 192 | 200 |

## Only known theorem before Euler:

Theorem (Eucl. IX.20). There are more prime numbers than any assigned number.




Proof. Suppose that $A$, $B$, $\Gamma$ are the only prime numbers.
Let $E\Delta$ be smallest number measured by all (i.e., their product).
Let $\Delta Z$ be the unity.
By hypothesis, $EZ$ is not a prime number.
So it must be measured by a prime number, say $H$.
But $H$ cannot be $A$, nor $B$, nor $\Gamma$. A contradiction.

Attention. It is not said that $p_1 p_2 p_3 \ldots p_k + 1$ is always prime.
Counter-example: $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509$.

# And then 2000 years nothing, ... until Euler E72 (1737)

The paper starts with a "proof" of Chr. Goldbach:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\ldots$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\ldots$$

we subtract the geom. series $\frac{1}{i-1}=\frac{1}{i}+\frac{1}{i^2}+\frac{1}{i^3}+\ldots$:

$$\not{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\ldots$$

$$=1+\not{\frac{1}{2}}+\frac{1}{3}+\not{\frac{1}{4}}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\not{\frac{1}{8}}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\ldots$$

$$\cancel{1}+\cancel{\frac{1}{2}}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\dots$$

$$=1+\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cancel{\frac{1}{8}}+\cancel{\frac{1}{9}}+\frac{1}{10}+\frac{1}{11}+\dots$$

$$\cancel{1}+\cancel{\frac{1}{2}}+\frac{1}{3}+\cancel{\frac{1}{4}}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\dots$$

$$=1+\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\cancel{\frac{1}{5}}+\frac{1}{6}+\frac{1}{7}+\cancel{\frac{1}{8}}+\cancel{\frac{1}{9}}+\frac{1}{10}+\frac{1}{11}+\dots$$

$$\cancel{1}+\cancel{\frac{1}{2}}+\frac{1}{3}+\cancel{\frac{1}{4}}+\cancel{\frac{1}{5}}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\dots$$

$$=1+\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\cancel{\frac{1}{5}}+\cancel{\frac{1}{6}}+\frac{1}{7}+\cancel{\frac{1}{8}}+\cancel{\frac{1}{9}}+\frac{1}{10}+\frac{1}{11}+\dots$$

and so on

$$\cancel{1}+\cancel{\frac{1}{2}}+\frac{1}{3}+\cancel{\frac{1}{4}}+\cancel{\frac{1}{5}}+\cancel{\frac{1}{6}}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\dots$$

$$=1+\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\cancel{\frac{1}{5}}+\cancel{\frac{1}{6}}+\cancel{\frac{1}{7}}+\cancel{\frac{1}{8}}+\cancel{\frac{1}{9}}+\frac{1}{10}+\frac{1}{11}+\dots$$

At the end we have the

**Theorema 1.**

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{31}+\frac{1}{35}+\ldots=1$$

*where the denominators are all powers* ($\geq 2$) *of all numbers* ($\geq 2$) *minus* 1.

Euler was fascinated by this result, because "this sort of series is completely different from the series which have been considered until now".

> "Les séries divergentes sont en général quelque chose de bien fatal, et c'est une honte qu'on ose y fonder aucune démonstration. On peut démontrer tout ce qu'on veut en les employant, et ce sont elles qui ont fait tant de malheurs et qui ont enfanté tant de paradoxes. Peut-on imaginer rien de plus horrible que de débiter $0 = 1 - 2^n + 3^n - 4^n + \ldots$, $n$ étant un nombre entier positif ?"
>
> (Letter of N. H. Abel to Holmboe, 16 jan. 1826)

Results concerning multiplication :

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\dots$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\dots$$

we subtract half of the series,

$$(1-\frac{1}{2})(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\dots)$$

$$=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}+\dots$$

we next subtract one third of the series,

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots\right)$$

$$=1+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\ldots$$

The next to subtract is one fifth ,

$$\left(1-\frac{1}{5}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\ldots\right)$$

$$=1+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\ldots$$

and so on ("tandem reperietur") until there is only 1 left to the right. We get :

## Theorema 7.

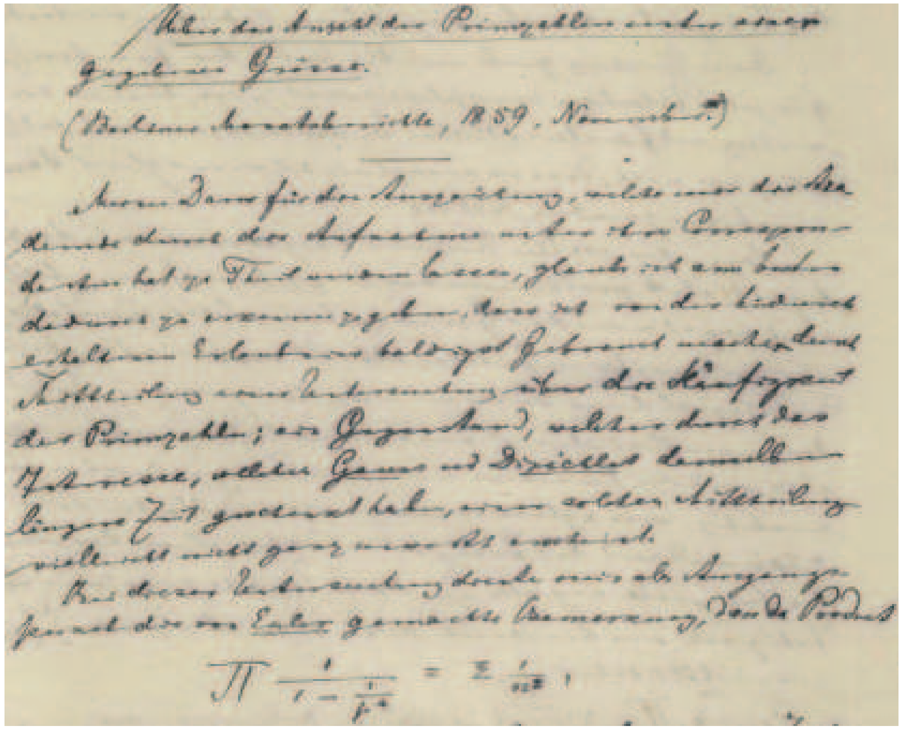
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots=$$
$$=\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ldots}$$

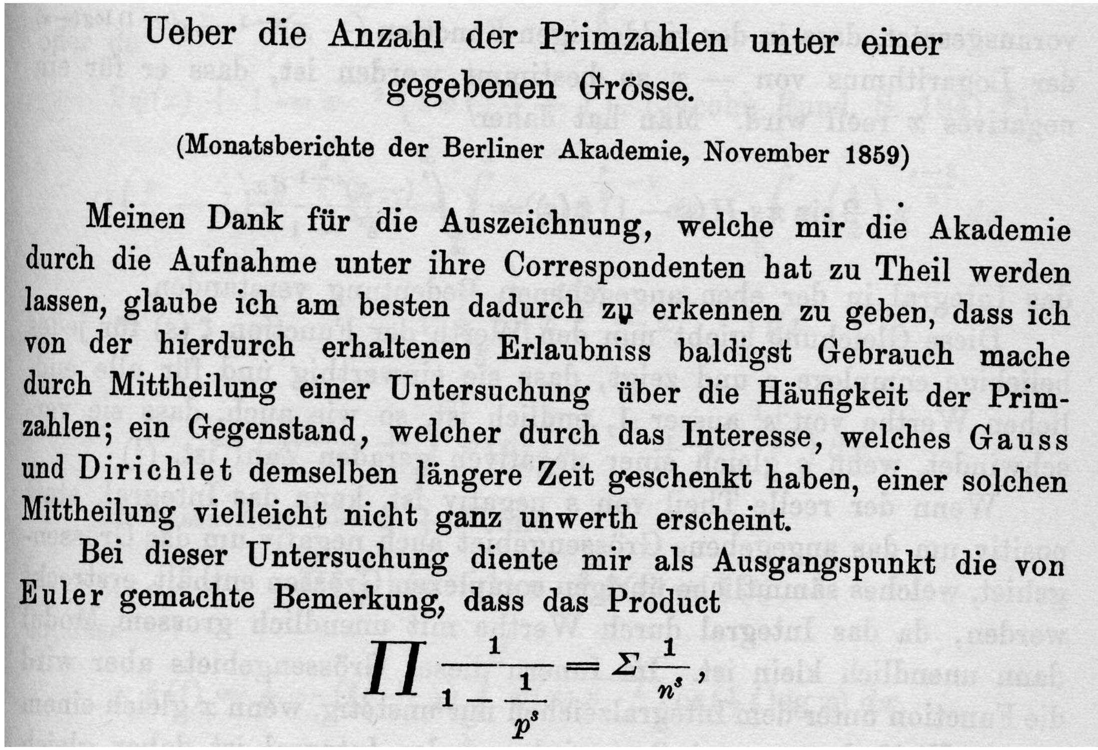
*where the denominator constituunt progressionem numerorum primorum* .

# Theorema 7.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots=$$
$$=\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ldots}$$

*where the denominator constituunt progressionem numerorum primorum* . $\Rightarrow$ Riemann (1859)





Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

Euler: Take logarithms

$$\ell(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots) = -\ell(1-\frac{1}{2})-\ell(1-\frac{1}{3})-\ell(1-\frac{1}{5})-\ldots$$

Apply $-\ell(1-\frac{1}{i}) = \frac{1}{i}+\frac{1}{2i^2}+\frac{1}{3i^3}+\ldots$:

**Theorema 19.** *Summa seriei reciprocae numerorum primorum est infinite magna*

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\ldots$$

$$= \ell(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots) = \ell(\ell\infty).$$

"One may well regard these investigations as marking the birth of analytic number-theory."

(A. Weil, *Number theory*, 1984, p. 267).

# VARIAE OBSERVATIONES CIRCA SERIES INFINITAS.

AVCTORE
*Leonh. Euler.*

## Theorema 19.

*Summa ſeriei reciprocae numerorum primorum*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

*eſt infinite magna; infinities tamen minor, quam ſumma ſeriei harmonicae* $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.}$ *Atque illius ſumma eſt huius ſummae quaſi logarithmus.*

## Demonſtratio.

Ponatur $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{etc.} = A$ atque $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = B$ et $\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \text{etc.} = C$. atque ita porro omnes poteſtates peculiaribus litteris deſignando; erit poſito $e$ pro numero cuius logarithmus hyperbolicus eſt 1

# Challenges: Fermat's Assertions.

Pierre de Fermat
(1601-1665)

"Mais voici ce que j'admire le plus : c'est que je suis quasi persuadé que tous les nombres progressifs augmentés de l'unité, desquels les exposants sont des nombres de la progression double, sont nombres premiers, comme

3 5 17 257 65537 4294967297 (. . .)

Je n'ai pas la démonstration exacte, mais j'ai exclu si grande quantité de diviseurs par démonstrations infaillibles, et j'ai de si grandes lumières, qui établissent ma pensée, que j'aurois peine à me dédire."

(Lettre de Fermat à Frenicle, août 1640)

Fermat affirme que tous les nombres de la forme $F_n = 2^{2^n} + 1$ sont premiers.

$$F_0 = 2^{2^0} + 1 = 3,\ F_1 = 2^{2^1} + 1 = 5,\ F_2 = 2^{2^2} + 1 = 17,\ F_3 = 2^{2^3} + 1 = 257,$$
$$F_4 = 2^{2^4} + 1 = 65537,\ F_5 = 2^{2^5} + 1 = 4294967297,\ \&c.$$

25 years old Euler found (E26) that Fermat was wrong and that

$$2^{2^5} + 1 = 2^{32} + 1 = 4294967297$$

"divisibilis est per 641".

He discovered this counter-example by proving (E134) that the only odd prime divisors of

$$a^{2^n} + b^{2^n}$$

must be of the form $2^{m+1}n + 1$.

**Amicable Numbers.** Pythagoreans discovered that:

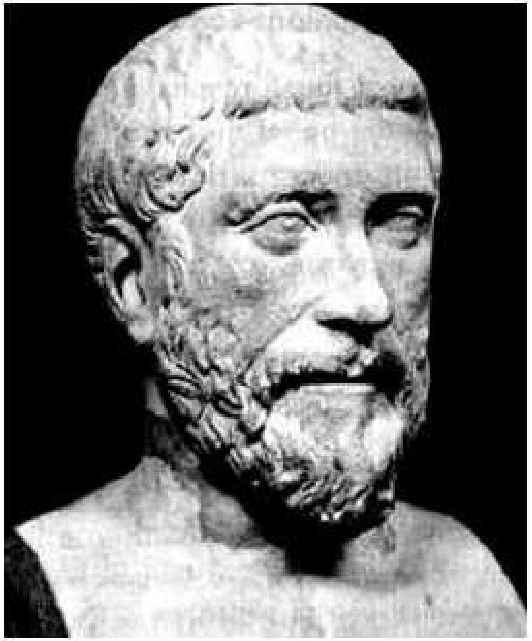
Sum of divisors of 220:

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

Sum of divisors of 284:

$$1 + 2 + 4 + 71 + 142 = 220 \; .$$

No more found by Greeks. ... Mersenne: Harmonie Universelle



220, 284

17'296, 18'416

9'363'584, 9'437'056

Euler (E152):

## Catalogus numerorum amicabilium.

I $\begin{cases} 2^2 \cdot 5 \cdot 11 \\ 2^2 \cdot 71 \end{cases}$

II $\begin{cases} 2^4 \cdot 23 \cdot 47 \\ 2^4 \cdot 1151 \end{cases}$

III $\begin{cases} 2^7 \cdot 191 \cdot 383 \\ 2^7 \cdot 73727 \end{cases}$

IV $\begin{cases} 2^2 \cdot 23 \cdot 5 \cdot 137 \\ 2^2 \cdot 23 \cdot 827 \end{cases}$

V $\begin{cases} 3^2 \cdot 7 \cdot 13 \cdot 5 \cdot 17 \\ 3^2 \cdot 7 \cdot 13 \cdot 107 \end{cases}$

VI $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 11 \cdot 19 \\ 3^2 \cdot 5 \cdot 13 \cdot 239 \end{cases}$

VII $\begin{cases} 3^2 \cdot 7^2 \cdot 13 \cdot 5 \cdot 41 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 251 \end{cases}$

VIII $\begin{cases} 3^2 \cdot 5 \cdot 7 \cdot 53 \cdot 1889 \\ 3^2 \cdot 5 \cdot 7 \cdot 102059 \end{cases}$

IX $\begin{cases} 2^2 \cdot 13 \cdot 17 \cdot 389 \cdot 509 \\ 2^2 \cdot 13 \cdot 17 \cdot 198899 \end{cases}$

X $\begin{cases} 3^2 \cdot 5 \cdot 19 \cdot 37 \cdot 7 \cdot 887 \\ 3^2 \cdot 5 \cdot 19 \cdot 37 \cdot 7103 \end{cases}$

XI $\begin{cases} 3^4 \cdot 5 \cdot 11 \cdot 29 \cdot 89 \\ 3^4 \cdot 5 \cdot 11 \cdot 2699 \end{cases}$

XII $\begin{cases} 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 41 \cdot 461 \\ 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19403 \end{cases}$

XIII $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 29 \cdot 569 \\ 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 17099 \end{cases}$

XIV $\begin{cases} 3^2 \cdot 7^2 \cdot 13 \cdot 97 \cdot 5 \cdot 193 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 97 \cdot 1163 \end{cases}$

XV $\begin{cases} 3^2 \cdot 7 \cdot 13 \cdot 41 \cdot 163 \cdot 5 \cdot 977 \\ 3^2 \cdot 7 \cdot 13 \cdot 41 \cdot 163 \cdot 5867 \end{cases}$

XVI $\begin{cases} 2^3 \cdot 17 \cdot 79 \\ 2^3 \cdot 23 \cdot 59 \end{cases}$

XVII $\begin{cases} 2^4 \cdot 23 \cdot 1367 \\ 2^4 \cdot 53 \cdot 607 \end{cases}$

XVIII $\begin{cases} 2^4 \cdot 47 \cdot 89 \\ 2^4 \cdot 53 \cdot 79 \end{cases}$

XIX $\begin{cases} 2^4 \cdot 23 \cdot 479 \\ 2^4 \cdot 89 \cdot 127 \end{cases}$

XX $\begin{cases} 2^4 \cdot 23 \cdot 467 \\ 2^4 \cdot 103 \cdot 107 \end{cases}$

XXI $\begin{cases} 2^4 \cdot 17 \cdot 5119 \\ 2^4 \cdot 239 \cdot 383 \end{cases}$

XXII $\begin{cases} 2^4 \cdot 17 \cdot 10303 \\ 2^4 \cdot 167 \cdot 1103 \end{cases}$

XXIII $\begin{cases} 2^4 \cdot 19 \cdot 1439 \\ 2^4 \cdot 149 \cdot 191 \end{cases}$

XXIV $\begin{cases} 2^5 \cdot 59 \cdot 1103 \\ 2^5 \cdot 79 \cdot 827 \end{cases}$

XXV $\begin{cases} 2^5 \cdot 37 \cdot 12671 \\ 2^5 \cdot 227 \cdot 2111 \end{cases}$

XXVI $\begin{cases} 2^5 \cdot 53 \cdot 10559 \\ 2^5 \cdot 79 \cdot 7127 \end{cases}$

XXVII $\begin{cases} 2^6 \cdot 79 \cdot 11087 \\ 2^6 \cdot 383 \cdot 2309 \end{cases}$

XXVIII $\begin{cases} 2^3 \cdot 383 \cdot 9203 \\ 2^3 \cdot 1151 \cdot 3067 \end{cases}$

XXIX $\begin{cases} 2^3 \cdot 11 \cdot 17 \cdot 263 \\ 2^3 \cdot 11 \cdot 43 \cdot 107 \end{cases}$

XXX $\begin{cases} 3^3 \cdot 5 \cdot 7 \cdot 71 \\ 3^3 \cdot 5 \cdot 17 \cdot 31 \end{cases}$

XXXI $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 29 \cdot 79 \\ 3^2 \cdot 5 \cdot 13 \cdot 11 \cdot 199 \end{cases}$

XXXII $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 47 \\ 3^2 \cdot 5 \cdot 13 \cdot 29 \cdot 31 \end{cases}$

XXXIII $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 37 \cdot 1583 \\ 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 227 \cdot 263 \end{cases}$

XXXIV $\begin{cases} 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 11 \cdot 220499 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 89 \cdot 29399 \end{cases}$

XXXV $\begin{cases} 3^2 \cdot 5 \cdot 19 \cdot 37 \cdot 47 \\ 3^2 \cdot 5 \cdot 19 \cdot 7 \cdot 227 \end{cases}$

XXXVI $\begin{cases} 2^4 \cdot 67 \cdot 37 \cdot 2411 \\ 2^4 \cdot 67 \cdot 227 \cdot 401 \end{cases}$

XXXVII $\begin{cases} 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 29 \\ 3^2 \cdot 5 \cdot 31 \cdot 89 \end{cases}$

XXXVIII $\begin{cases} 2 \cdot 5 \cdot 23 \cdot 29 \cdot 673 \\ 2 \cdot 5 \cdot 7 \cdot 60659 \end{cases}$

XXXIX $\begin{cases} 2 \cdot 5 \cdot 7 \cdot 19 \cdot 107 \\ 2 \cdot 5 \cdot 47 \cdot 359 \end{cases}$

XL $\begin{cases} 2^3 \cdot 11 \cdot 163 \cdot 191 \\ 2^3 \cdot 31 \cdot 11807 \end{cases}$

XLI $\begin{cases} 3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 11 \cdot 19 \cdot 367 \\ 3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 79 \cdot 1103 \end{cases}$

XLII $\begin{cases} 3^2 \cdot 5 \cdot 23 \cdot 11 \cdot 19 \cdot 367 \\ 3^2 \cdot 5 \cdot 23 \cdot 79 \cdot 1103 \end{cases}$

XLIII $\begin{cases} 2^3 \cdot 11 \cdot 59 \cdot 173 \\ 2^3 \cdot 57 \cdot 2609 \end{cases}$

XLIV $\begin{cases} 2^3 \cdot 11 \cdot 23 \cdot 2543 \\ 2^3 \cdot 383 \cdot 1907 \end{cases}$

XLV $\begin{cases} 2^3 \cdot 11 \cdot 23 \cdot 1871 \\ 2^3 \cdot 467 \cdot 1151 \end{cases}$

XLVI $\begin{cases} 2^3 \cdot 11 \cdot 23 \cdot 1619 \\ 2 \cdot 719 \cdot 647 \end{cases}$

XLVII $\begin{cases} 2^3 \cdot 11 \cdot 29 \cdot 239 \\ 2^3 \cdot 191 \cdot 449 \end{cases}$

XLVIII $\begin{cases} 2^3 \cdot 29 \cdot 47 \cdot 59 \\ 2^3 \cdot 17 \cdot 4799 \end{cases}$

XLIX $\begin{cases} 2^4 \cdot 17 \cdot 167 \cdot 13679 \\ 2^4 \cdot 809 \cdot 51071 \end{cases}$

L $\begin{cases} 2^4 \cdot 23 \cdot 47 \cdot 9767 \\ 2^4 \cdot 1583 \cdot 7103 \end{cases}$

LI $\begin{cases} 2^2 \cdot 5 \cdot 13 \cdot 1187 \\ 2^2 \cdot 43 \cdot 2267 \end{cases}$

LII $\begin{cases} 3^2 \cdot 7 \cdot 13 \cdot 5 \cdot 17 \cdot 1187 \\ 3^2 \cdot 7 \cdot 13 \cdot 131 \cdot 971 \end{cases}$

LIII $\begin{cases} 3^2 \cdot 7^2 \cdot 13 \cdot 53 \cdot 11 \cdot 211 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 53 \cdot 2543 \end{cases}$

LIV $\begin{cases} 3^2 \cdot 5^2 \cdot 11 \cdot 59 \cdot 179 \\ 3^2 \cdot 5^2 \cdot 17 \cdot 19 \cdot 359 \end{cases}$

LV $\begin{cases} 3^3 \cdot 5 \cdot 17 \cdot 23 \cdot 397 \\ 3^3 \cdot 5 \cdot 7 \cdot 21491 \end{cases}$

LVI $\begin{cases} 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 47 \cdot 7019 \\ 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 389 \cdot 863 \end{cases}$

LVII $\begin{cases} 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 53 \cdot 6959 \\ 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 179 \cdot 2087 \end{cases}$

LVIII $\begin{cases} 3^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 47 \cdot 7019 \\ 3^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 389 \cdot 863 \end{cases}$

LIX $\begin{cases} 3^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 53 \cdot 6959 \\ 3^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 179 \cdot 2087 \end{cases}$

His adjicere lubet duo paria sequentia, quæ sunt formæ diversæ a præcedentibus:

LX $\begin{cases} 2^2 \cdot 19 \cdot 41 \\ 2^5 \cdot 199 \end{cases}$

LXI $\begin{cases} 2^2 \cdot 41 \cdot 467 \\ 2^5 \cdot 19 \cdot 233 \end{cases}$

## The "Great Fermat":

$$a^n + b^n = c^n \quad \text{impossible for } n > 2.$$

Euler (E98) elaborates proof of Fermat-Frenicle for $n = 4$ (the only idea of proof which Fermat allowed to escape!!) as follows:

**Lemma 2.** $a^2 + b^2 = \text{quadr.} \Rightarrow \begin{array}{l} a = p^2 - q^2 \\ b = 2pq \end{array}$

**Lemma 3.** $a^2 - b^2 = \text{quadr.} \Rightarrow \begin{array}{l} a = p^2 + q^2 \\ b = p^2 - q^2 \text{ or } 2pq\ . \end{array}$

Now suppose $a^4 + b^4 = \text{quadr.}$.

Lemma 2: $\Rightarrow a^2 = p^2 - q^2, \quad b^2 = 2pq \quad \Rightarrow 2q, p = \text{quadr.}$

Lemma 3: $\Rightarrow p = m^2 + n^2 \quad q = 2mn \quad \Rightarrow 4mn, mn = \text{quadr.}$

$\Rightarrow m = x^2, n = y^2, \ p = x^4 + y^4 = \text{quadr.}$ "descente infinie"....

## Subsequent Theorems of E98:

**Theorema 2.** $a^4 - b^4 \neq$ quadr.

**Theorema 3.** $2a^4 + 2b^4 \neq$ quadr.

**Theorema 4.** $2a^4 - 2b^4 \neq$ quadr.

**Theorema 5.**
$ma^4 - m^3b^4 \neq$ quadr., $2ma^4 - 2m^3b^4 \neq$ quadr.

**Theorema 6.**
$ma^4 + m^3b^4 \neq$ quadr., $2ma^4 + 2m^3b^4 \neq$ quadr.

**Theorema 7.** (Fermatianum).

Triangular numbers $\frac{x(x+1)}{2} \neq$ **bi**quadr.

**Theorema 8.** $a^4 + 2b^4 \neq$ quadr.

**Theorema 10.**
$a^3 + 1 = c^2$ impossible except for $a = 2, \ c = 3$.

... just to have an idea of Euler's working power...

## **The “Great Fermat”** for $n = 3$:

> “Ich habe nun wohl Demonstrationen gefunden, dass $a^3 + b^3 \neq c^3$ und $a^4 + b^4 \neq c^4$, wo $\neq$ unmöglich gleich bedeutet. Aber die Demonstrationen für diese zwei casus sind so voneinander verschieden, dass ich keine Möglichkeit sehe, daraus eine allgemeine Demonstration für $a^n + b^n \neq c^n$ si $n > 2$ herzuleiten.”
>
> (First mention of proof for $n = 3$, letter of Euler to Goldbach, Berlin 4. Aug. 1753.)

finally published as very last theorem of Anleitung zur Algebra (E388, 1770).

$n = 5$: Dirichlet (1824) , $n = 14$: Dirichlet, ......

all $n$: read newspapers of 1994....

E255: found interesting counter-examples

$$3^3 + 4^3 + 5^3 = 6^3, \;\; 1^3 + 6^3 + 8^3 = 9^3, \;\; 1^3 + 12^3 = 9^3 + 10^3.$$

# Diophant-Bachet: Every number is sum of 4 squares. Prove!!

Joseph-Louis Lagrange
(1736-1813)

Pendant des décennies, la correspondance entre Euler et Goldbach tourne autour de ce problème.
Ce n'est qu'en 1773 qu'Euler obtient une preuve, inspirée de celle de Lagrange (1770) .

$1 = 1^2$
$2 = 1^2 + 1^2$
$3 = 1^2 + 1^2 + 1^2$
$4 = 2^2$
$5 = 2^2 + 1^2$
$6 = 2^2 + 1^2 + 1^2$
$7 = 2^2 + 1^2 + 1^2 + 1^2$
$8 = 2^2 + 2^2$
$9 = 3^2$
$10 = 3^2 + 1^2$
$11 = 3^2 + 1^2 + 1^2$
$12 = 2^2 + 2^2 + 2^2$
$13 = 3^2 + 2^2$
$14 = 3^2 + 2^2 + 1^2$
$15 = 3^2 + 2^2 + 1^2 + 1^2$
$16 = 4^2$
$17 = 4^2 + 1^2$
$18 = 3^2 + 3^2$
$19 = 3^2 + 3^2 + 1^2$
$20 = 4^2 + 2^2$
$21 = 4^2 + 2^2 + 1^2$
$22 = 3^2 + 3^2 + 2^2$
$23 = 3^2 + 3^2 + 2^2 + 1^2$
$24 = 4^2 + 2^2 + 2^2$
$25 = 5^2$
$26 = 5^2 + 1^2$
$27 = 5^2 + 1^2 + 1^2$
$28 = 3^2 + 3^2 + 3^2 + 1^2$
$29 = 5^2 + 2^2$
$30 = 5^2 + 2^2 + 1^2$
$31 = 3^2 + 3^2 + 3^2 + 2^2$
$32 = 4^2 + 4^2$
$33 = 5^2 + 2^2 + 2^2$
$34 = 5^2 + 3^2$
$35 = 5^2 + 3^2 + 1^2$
$36 = 6^2$
$37 = 6^2 + 1^2$
$38 = 6^2 + 1^2 + 1^2$
$39 = 5^2 + 3^2 + 2^2 + 1^2$
$40 = 6^2 + 2^2$
$41 = 5^2 + 4^2$
$42 = 5^2 + 4^2 + 1^2$
$43 = 5^2 + 3^2 + 3^2$
$44 = 6^2 + 2^2 + 2^2$
$45 = 6^2 + 3^2$
$46 = 6^2 + 3^2 + 1^2$
$47 = 5^2 + 3^2 + 3^2 + 2^2$
$48 = 4^2 + 4^2 + 4^2$
$49 = 7^2$
$50 = 5^2 + 5^2$
$51 = 7^2 + 1^2 + 1^2$
$52 = 6^2 + 4^2$
$53 = 7^2 + 2^2$
$54 = 7^2 + 2^2 + 1^2$
$55 = 5^2 + 5^2 + 2^2 + 1^2$
$56 = 6^2 + 4^2 + 2^2$
$57 = 7^2 + 2^2 + 2^2$
$58 = 7^2 + 3^2$
$59 = 7^2 + 3^2 + 1^2$
$60 = 5^2 + 5^2 + 3^2 + 1^2$
$61 = 6^2 + 5^2$
$62 = 7^2 + 3^2 + 2^2$
$63 = 5^2 + 5^2 + 3^2 + 2^2$
$64 = 8^2$
$65 = 7^2 + 4^2$
$66 = 8^2 + 1^2 + 1^2$
$67 = 7^2 + 3^2 + 3^2$
$68 = 8^2 + 2^2$
$69 = 8^2 + 2^2 + 1^2$
$70 = 6^2 + 5^2 + 3^2$
$71 = 6^2 + 5^2 + 3^2 + 1^2$
$72 = 6^2 + 6^2$
$73 = 8^2 + 3^2$
$74 = 7^2 + 5^2$
$75 = 7^2 + 5^2 + 1^2$
$76 = 6^2 + 6^2 + 2^2$
$77 = 8^2 + 3^2 + 2^2$
$78 = 7^2 + 5^2 + 2^2$
$79 = 5^2 + 5^2 + 5^2 + 2^2$
$80 = 8^2 + 4^2$
$81 = 9^2$
$82 = 9^2 + 1^2$
$83 = 9^2 + 1^2 + 1^2$
$84 = 8^2 + 4^2 + 2^2$
$85 = 7^2 + 6^2$
$86 = 9^2 + 2^2 + 1^2$
$87 = 6^2 + 5^2 + 5^2 + 1^2$
$88 = 6^2 + 6^2 + 4^2$
$89 = 8^2 + 5^2$
$90 = 9^2 + 3^2$
$91 = 9^2 + 3^2 + 1^2$
$92 = 6^2 + 6^2 + 4^2 + 2^2$
$93 = 8^2 + 5^2 + 2^2$
$94 = 9^2 + 3^2 + 2^2$
$95 = 6^2 + 5^2 + 5^2 + 3^2$
$96 = 8^2 + 4^2 + 4^2$
$97 = 9^2 + 4^2$
$98 = 7^2 + 7^2$
$99 = 9^2 + 3^2 + 3^2$

# Easier: Numbers sum of 2 squares: (E228 + E241)

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 9 | 17 | 25 | 33 | 41 | 49 | 57 | 65 | 73 | 81 | 89 | 97 | 105 | 113 | 121 | 129 | 137 | 145 | 153 | 161 | 169 | 177 | 185 | 193 |
| 2 | 10 | 18 | 26 | 34 | 42 | 50 | 58 | 66 | 74 | 82 | 90 | 98 | 106 | 114 | 122 | 130 | 138 | 146 | 154 | 162 | 170 | 178 | 186 | 194 |
| 3 | 11 | 19 | 27 | 35 | 43 | 51 | 59 | 67 | 75 | 83 | 91 | 99 | 107 | 115 | 123 | 131 | 139 | 147 | 155 | 163 | 171 | 179 | 187 | 195 |
| 4 | 12 | 20 | 28 | 36 | 44 | 52 | 60 | 68 | 76 | 84 | 92 | 100 | 108 | 116 | 124 | 132 | 140 | 148 | 156 | 164 | 172 | 180 | 188 | 196 |
| 5 | 13 | 21 | 29 | 37 | 45 | 53 | 61 | 69 | 77 | 85 | 93 | 101 | 109 | 117 | 125 | 133 | 141 | 149 | 157 | 165 | 173 | 181 | 189 | 197 |
| 6 | 14 | 22 | 30 | 38 | 46 | 54 | 62 | 70 | 78 | 86 | 94 | 102 | 110 | 118 | 126 | 134 | 142 | 150 | 158 | 166 | 174 | 182 | 190 | 198 |
| 7 | 15 | 23 | 31 | 39 | 47 | 55 | 63 | 71 | 79 | 87 | 95 | 103 | 111 | 119 | 127 | 135 | 143 | 151 | 159 | 167 | 175 | 183 | 191 | 199 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 | 88 | 96 | 104 | 112 | 120 | 128 | 136 | 144 | 152 | 160 | 168 | 176 | 184 | 192 | 200 |

"red"= prime, "blue"= sum 2 squ., "violet"= both

Observation:

There is no "red" in $= 4n + 1$!

Theorem: Every prime of type $4n + 1$ is sum of two squares.

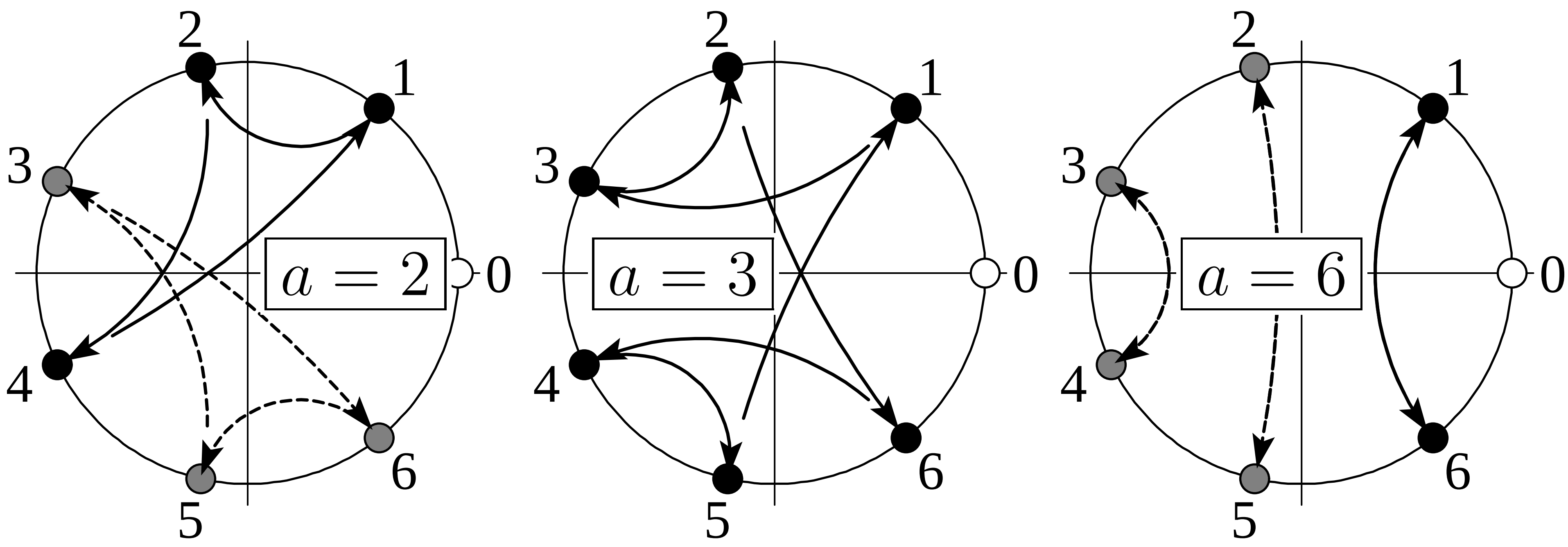
**Power Residuals.** (E262).

"residua ex divisione potestatum relicta...":

Let $p$ be a prime number, say, $p = 7$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Progressio geometrica | 1, | 2, | 4, | 8, | 16, | 32, | 64, | ... |
| Residua | 1, | 2, | 4, | 1, | 2, | 4, | 1, | ... |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Progressio geometrica | 1, | 3, | 9, | 27, | 81, | 243, | 729, | ... |
| Residua | 1, | 3, | 2, | 6, | 4, | 5, | 1, | ... |



## The "Little" Fermat. (E134, E262, E271, E449)

- Results depend only on preceeding residual;
- $\Rightarrow$ periodic sequence.
- Either all $p-1$ restes appear ("primitive root")
- ... or $\frac{p-1}{2}$ restes appear, or $\frac{p-1}{3}$, etc. We conclude:

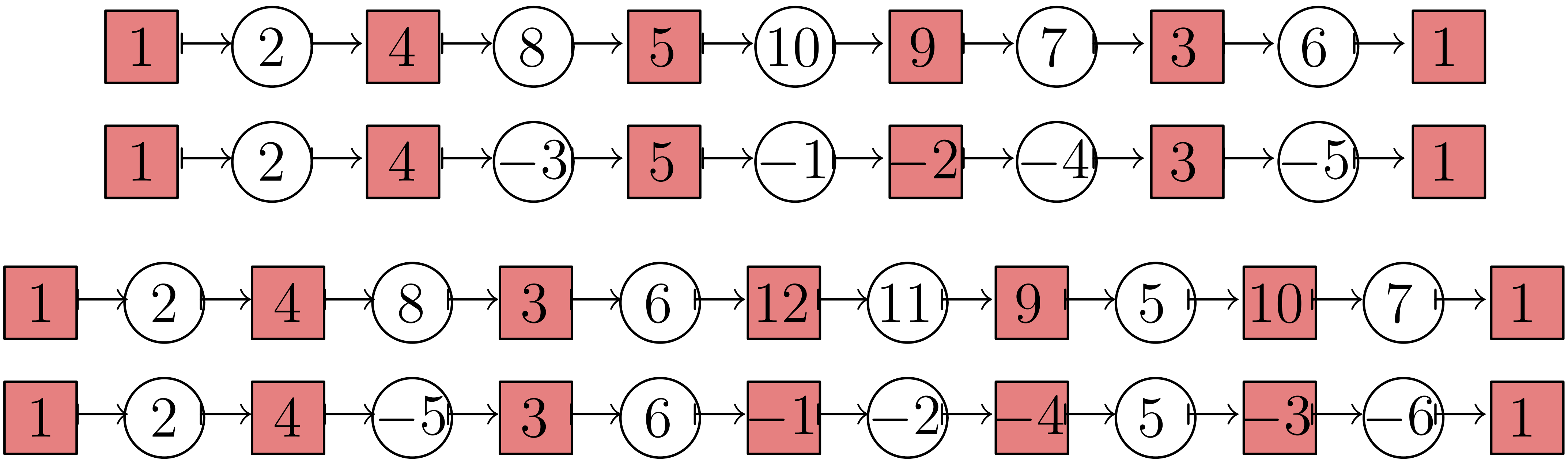
**Theorem.**

$$a^{p-1} = 1 \pmod{p} .$$

**Quadratic Residuals.** (E552).

Question. Which residuals are residuals of a square?

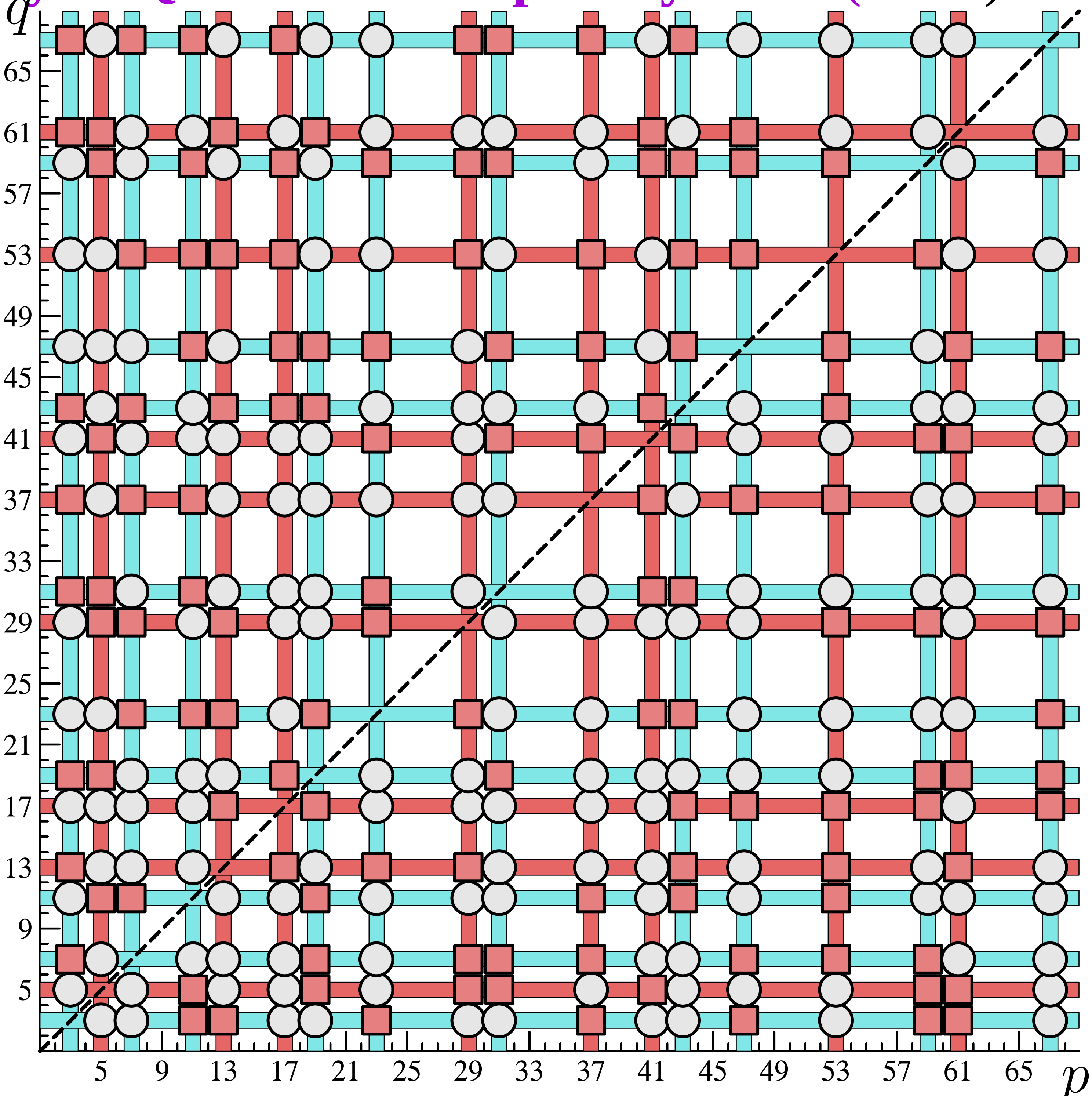
Answer: Take a primitive root and look ($a = 2$ and $p = 11, 13$):




ConclusionThe rest $-1$ is a quadratic residual if $p = 4n + 1$.

Or: There is a $k$ such that $p = 4n + 1$ divides $k^2 + 1$ (i.e., a sum of two squares). This is ingedient of proof of E228.

# Discovery of Quadratic Reciprocity Law. (E552).



# Grazie