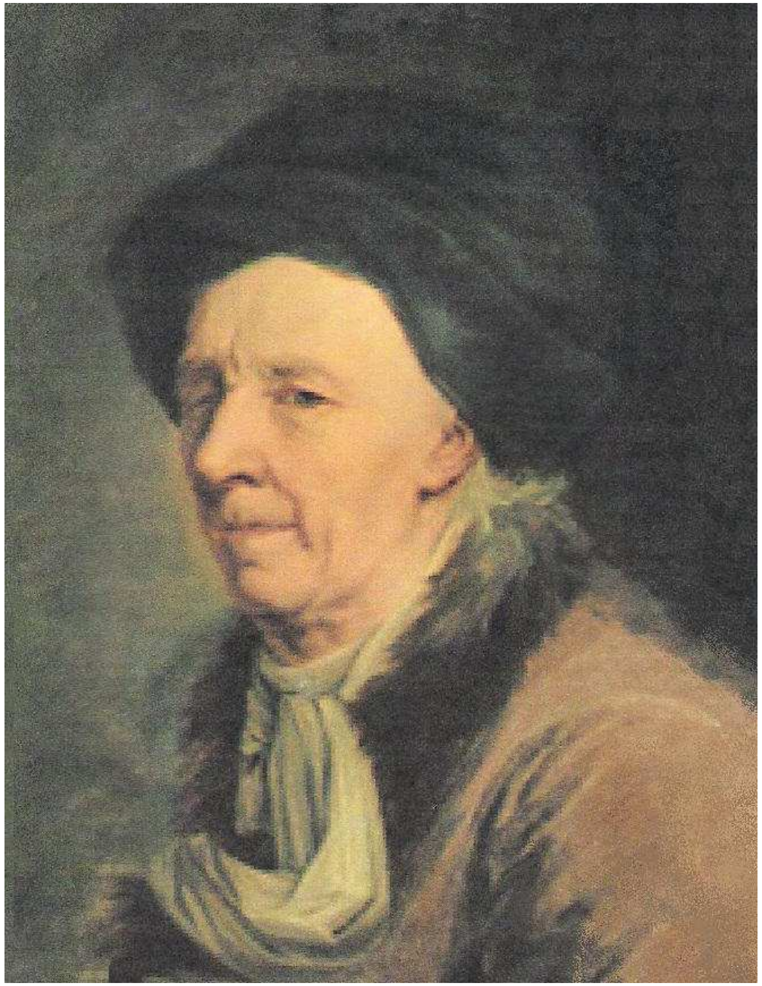
From Euler to Fourier, MP3 and JPEG

From Euler to Fourier, MP3 and JPEG

1707  1783

Ph. Henry & G. Wanner

I. The Quadrature of the Circle.

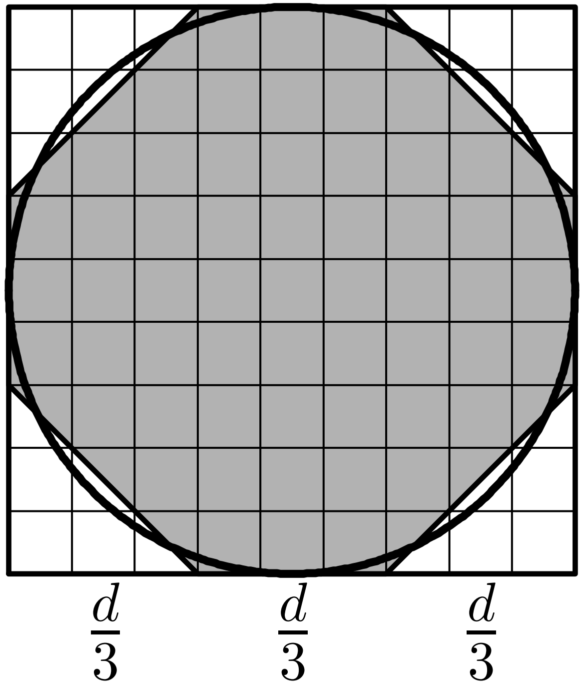
Mesopotamia 5000 B.C.

Circle = oldest mathematical figure

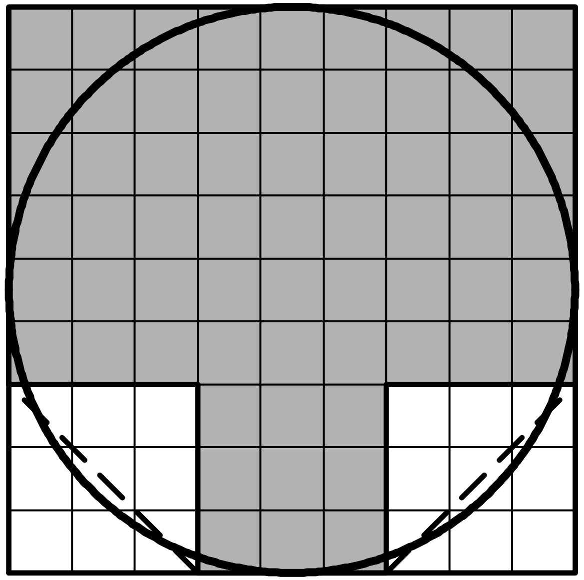
Papyrus Rhind, Egypt 1650 B.C.

Papyrus Rhind, Problem 50: Find Area of the Circle !

Papyrus Rhind



$$\mathcal{A}_{\text{square}} = 81 = 9^2$$



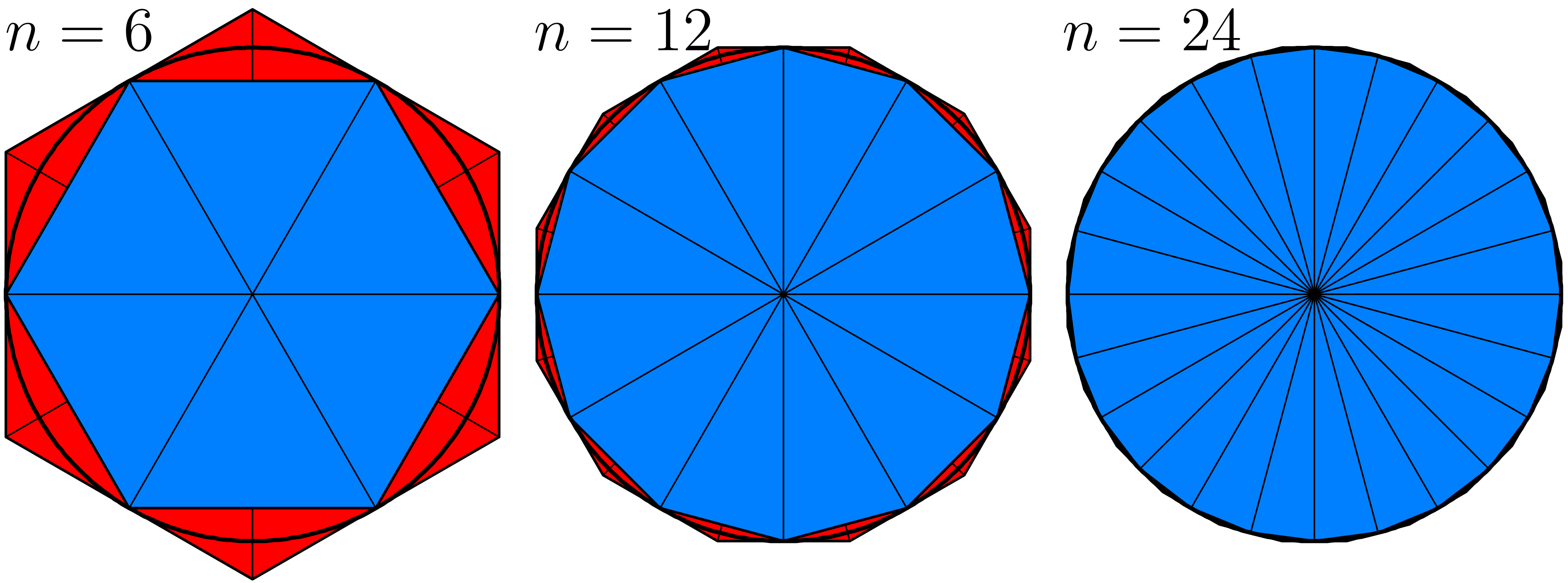
$$\mathcal{A}_{\text{circle}} = 81 - 18 = 63 \approx 8^2$$

Result:

Area of circle of diameter d $=$ Area of square of side $\frac{8}{9}d$

Critics: Is only approximation; is it too large? too small? how precise ??

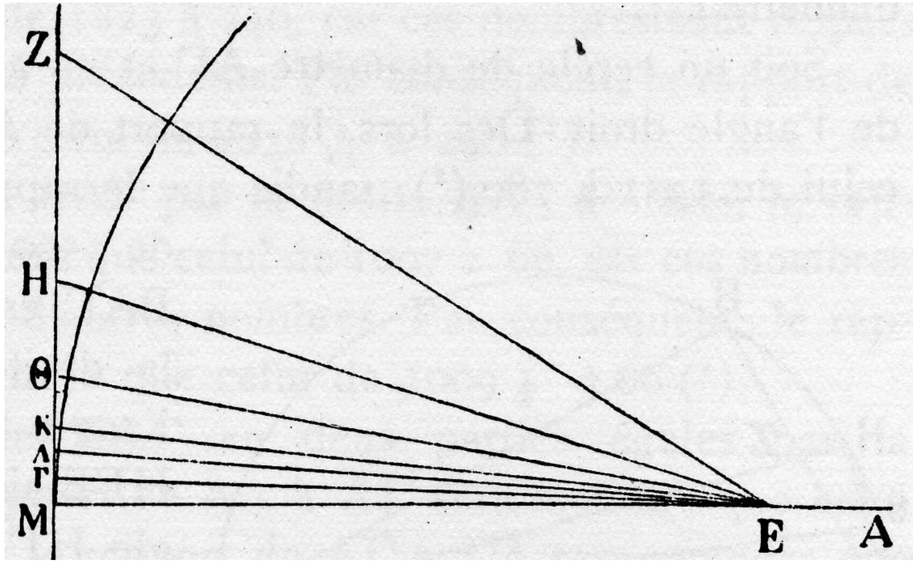
Archimedes (≈ 250 B.C.) Greek Rigour and Performance:



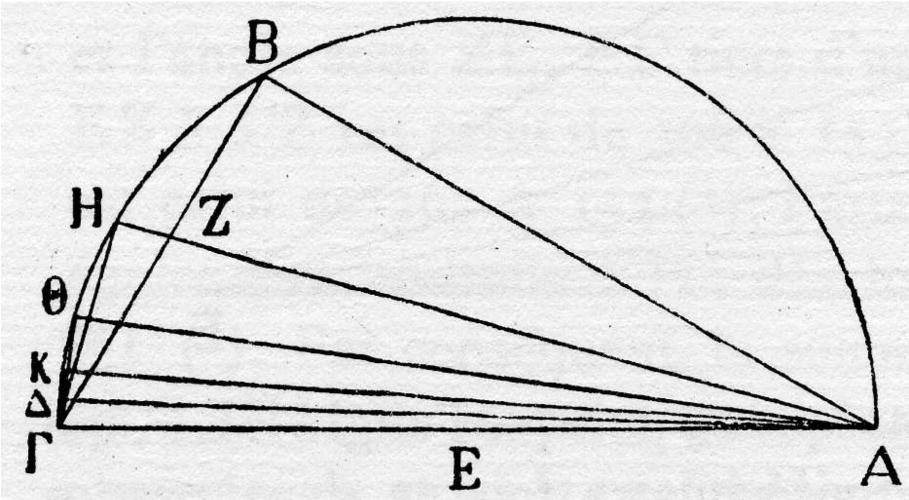
Archimedes' Proposition 3:

$$d \cdot 3\frac{10}{71} < \text{Perimeter of circle} < d \cdot 3\frac{1}{7}$$

For the proof, Archimedes computed areas and perimeters of circumscribed and inscribed n-gons for $n = 6, 12, 24, 48, 96$.



circumscribed



inscribed n-gon

Details of Archimedes' Computations

(as reconstructed by Ver Eecke 1917)

1. Par hypothèse : angle $\mathrm{ZE\Gamma} = \frac{1}{3}$ angle droit, d'où angle $\Gamma\mathrm{ZE} = \frac{2}{3}$ angle droit, d'où : $\mathrm{EZ} = 2\,\Gamma\mathrm{Z}$, d'où : $\frac{\mathrm{EZ}}{\Gamma\mathrm{Z}} = \frac{2}{1} = \frac{306}{153}$. Or, $\mathrm{E}\Gamma = \sqrt{\overline{\mathrm{EZ}}^2 - \overline{\Gamma\mathrm{Z}}^2} = \sqrt{306^2 - 153^2} = \sqrt{70227} > 265$, d'où : $\mathrm{E}\Gamma > 265$. Donc : $\frac{\mathrm{E}\Gamma}{\Gamma\mathrm{Z}} > \frac{265}{153}$, ou, suivant le texte, $\frac{\mathrm{E}\Gamma}{\Gamma\mathrm{Z}} = \frac{265}{153}$ en valeur approchée.

2. Cf. Euclide, livre VI, proposition 3.

3. Puisque $\frac{\mathrm{ZE}}{\mathrm{E}\Gamma} = \frac{\mathrm{ZH}}{\mathrm{H}\Gamma}$, on a $\frac{\mathrm{ZE}+\mathrm{E}\Gamma}{\mathrm{E}\Gamma} = \frac{\mathrm{ZH}+\mathrm{H}\Gamma}{\mathrm{H}\Gamma}$, ou $\frac{\mathrm{ZE}+\mathrm{E}\Gamma}{\mathrm{E}\Gamma} = \frac{\mathrm{Z}\Gamma}{\mathrm{H}\Gamma}$, d'où, comme le texte : $\frac{\mathrm{ZE}+\mathrm{E}\Gamma}{\mathrm{Z}\Gamma} = \frac{\mathrm{E}\Gamma}{\Gamma\mathrm{H}}$. Or, $\Gamma\mathrm{Z} = 153$, $\mathrm{EZ} = 306$ et $\mathrm{E}\Gamma > 265$; donc $\frac{\mathrm{E}\Gamma}{\Gamma\mathrm{H}} > \frac{306+265}{153}$, ou $\frac{\mathrm{E}\Gamma}{\Gamma\mathrm{H}} > \frac{571}{153}$, d'où $\frac{\overline{\mathrm{E}\Gamma}^2 + \overline{\Gamma\mathrm{H}}^2}{\overline{\Gamma\mathrm{H}}^2} > \frac{571^2 + 153^2}{153^2}$, ou $\frac{\overline{\mathrm{EH}}^2}{\overline{\Gamma\mathrm{H}}^2} > \frac{349450}{23409}$, d'où, comme le texte : $\frac{\mathrm{EH}}{\Gamma\mathrm{H}} > \frac{591\frac{1}{8}}{153}$.

4. On aura : $\frac{\mathrm{NE}}{\mathrm{E}\Gamma} = \frac{\mathrm{H}\Theta}{\Theta\Gamma}$, d'où : $\frac{\mathrm{HE}+\mathrm{E}\Gamma}{\mathrm{E}\Gamma} = \frac{\mathrm{H}\Theta+\Theta\Gamma}{\Theta\Gamma}$, d'où : $\frac{\mathrm{HE}+\mathrm{E}\Gamma}{\mathrm{H}\Theta+\Theta\Gamma} = \frac{\mathrm{E}\Gamma}{\Theta\Gamma}$, ou $\frac{\mathrm{HE}+\mathrm{E}\Gamma}{\mathrm{H}\Gamma} = \frac{\mathrm{E}\Gamma}{\Theta\Gamma}$. Or, $\frac{\mathrm{E}\Gamma}{\Gamma\mathrm{H}} > \frac{571}{153}$ et $\frac{\mathrm{EH}}{\Gamma\mathrm{H}} > \frac{591\frac{1}{8}}{153}$, d'où, substituant, $\frac{\mathrm{E}\Gamma}{\Theta\Gamma} > \frac{591\frac{1}{8} + 571}{153}$ ou, comme le texte : $\frac{\mathrm{E}\Gamma}{\Theta\Gamma} > \frac{1162\frac{1}{8}}{153}$, d'où : $\frac{\overline{\mathrm{E}\Gamma}^2}{\overline{\Theta\Gamma}^2} > \frac{(1162\frac{1}{8})^2}{153^2}$, d'où : $\frac{\overline{\mathrm{E}\Gamma}^2 + \overline{\Theta\Gamma}^2}{\overline{\Theta\Gamma}^2} > \frac{(1162\frac{1}{8})^2 + 153^2}{153^2}$, ou $\frac{\overline{\Theta\mathrm{E}}^2}{\overline{\Theta\Gamma}^2} > \frac{1350534\frac{33}{64} + 23409}{23409}$, d'où : $\frac{\Theta\mathrm{E}}{\Theta\Gamma} > \frac{1172\frac{1}{8}}{153}$.

1. On aura : $\frac{\Theta E}{E\Gamma} = \frac{\Theta K}{K\Gamma}$, d'où : $\frac{\Theta E + E\Gamma}{\Theta K + K\Gamma} = \frac{E\Gamma}{K\Gamma}$, ou $\frac{\Theta E + E\Gamma}{\Theta\Gamma} = \frac{E\Gamma}{K\Gamma}$. Or, $\frac{\Theta E}{\Theta\Gamma} > \frac{1172\frac{1}{8}}{153}$, et $\frac{E\Gamma}{\Theta\Gamma} > \frac{1162\frac{1}{8}}{153}$ d'où : $\frac{E\Gamma}{K\Gamma} > \frac{1172\frac{1}{8} + 1162\frac{1}{8}}{153}$, ou $\frac{E\Gamma}{K\Gamma} > \frac{2334\frac{1}{4}}{153}$, d'où :
$\frac{\overline{E\Gamma}^2 + \overline{K\Gamma}^2}{K\Gamma^2} > \frac{(2334\frac{1}{4})^2 + 153^2}{153^2}$, ou $\frac{\overline{EK}^2}{\overline{K\Gamma}^2} > \frac{5472132\frac{1}{16}}{23409}$, d'où, comme le texte : $\frac{EK}{\Gamma K} > \frac{2339\frac{1}{4}}{153}$.

2. On aura: $\frac{KE}{E\Gamma} = \frac{K\Lambda}{\Lambda\Gamma}$, d'où : $\frac{KE + E\Gamma}{K\Lambda + \Lambda\Gamma} = \frac{E\Gamma}{\Lambda\Gamma}$, ou $\frac{KE + E\Gamma}{K\Gamma} = \frac{E\Gamma}{\Lambda\Gamma}$. Or, $\frac{KE}{K\Gamma} > \frac{2339\frac{1}{4}}{153}$, et $\frac{E\Gamma}{K\Gamma} > \frac{2334\frac{1}{4}}{153}$, d'où : $\frac{E\Gamma}{\Lambda\Gamma} > \frac{2339\frac{1}{4} + 2334\frac{1}{4}}{153}$, ou $\frac{E\Gamma}{\Lambda\Gamma} > \frac{4673\frac{1}{2}}{153}$.

1. Reprenons plus explicitement le texte qui précède au point où nous l'avons laissé dans la note avant-précédente :

On a : $A\Gamma = 2\,E\Gamma$ et $\Lambda M = 2\,\Gamma\Lambda$, d'où, par substitution dans la relation $\frac{E\Gamma}{\Gamma\Lambda} > \frac{4673\frac{1}{2}}{153}$, il vient : $\frac{A\Gamma}{\Lambda M} > \frac{4673\frac{1}{2}}{153}$, d'où : $\frac{A\Gamma}{96 \times \Lambda M} > \frac{4673\frac{1}{2}}{96 \times 153}$, ou, suivant le texte : $\frac{A\Gamma}{\text{polygone de 96 côtés}} > \frac{4673\frac{1}{2}}{14688}$; or, on a $\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$, tandis que $667\frac{1}{2} < \frac{1}{7} 4673\frac{1}{2}$. Donc, $\frac{\text{polygone circonscrit}}{A\Gamma = \text{diam. cercle}} < 3 + \frac{1}{7}$, d'où polygone circonscrit $< 3 + \frac{1}{7}$ diam. cercle, d'où, en vertu de la prop. I du livre I de la *Sphère et du Cylindre*, à fortiori, circonférence cercle $< 3 + \frac{1}{7}$ diamètre cercle.

2. Puisque l'angle $BA\Gamma = \frac{1}{3}$ angle droit, $B\Gamma$ est le côté de l'hexagone inscrit et l'on a : $\frac{AB}{B\Gamma} = \frac{\sqrt{3}}{1}$. Or, $3 \times 608400 = 1825200 < 1825201$, d'où : $\frac{3}{1} < \frac{1825201}{608400}$, d'où : $\frac{\sqrt{3}}{1} < \frac{1351}{780}$, d'où, comme le texte : $\frac{AB}{B\Gamma} < \frac{1351}{780}$.

3. $\frac{A\Gamma}{\Gamma B} = \frac{2}{1} = \frac{2 \times 780}{780} = \frac{1560}{780}$.

4. Les triangles $AH\Gamma$, ΓHZ sont équiangles, car les angles $H\Gamma B$, $HA\Gamma$ sont égaux comme inscrits et mesurés par des arcs égaux, et l'angle droit en H est commun, donc : $\frac{AH}{H\Gamma} = \frac{\Gamma H}{HZ} = \frac{A\Gamma}{\Gamma Z}$. Or, AZ divise l'arc ΓB en deux parties égales, d'où : $\frac{A\Gamma}{AB} = \frac{\Gamma Z}{ZB}$,

d'où : $\frac{A\Gamma}{A\Gamma + AB} = \frac{\Gamma Z}{\Gamma Z + ZB}$, ou $\frac{A\Gamma}{\Gamma Z} = \frac{A\Gamma + AB}{B\Gamma}$, d'où : $\frac{AH}{H\Gamma} = \frac{A\Gamma + AB}{B\Gamma} = \frac{A\Gamma}{B\Gamma} + \frac{AB}{B\Gamma}$. Or, $\frac{A\Gamma}{B\Gamma} = \frac{1560}{780}$, et $\frac{AB}{B\Gamma} < \frac{1351}{780}$; donc $\frac{AH}{H\Gamma} < \frac{1560 + 1351}{780}$, ou $\frac{AH}{H\Gamma} < \frac{2911}{780}$. D'autre part, $\frac{\overline{AH}^2}{\overline{H\Gamma}^2} < \frac{(2911)^2}{(780)^2}$, d'où : $\frac{\overline{AH}^2 + \overline{H\Gamma}^2}{\overline{H\Gamma}^2} < \frac{2911^2 + 780^2}{780^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{H\Gamma}^2} < \frac{9082321}{608400}$, d'où, comme le texte : $\frac{A\Gamma}{H\Gamma} < \frac{3013\frac{3}{4}}{780}$.

1. On aura, comme dans le cas précédent : $\frac{A\Theta}{\Theta\Gamma} = \frac{A\Gamma + AH}{H\Gamma} = \frac{A\Gamma}{HA} + \frac{AH}{H\Gamma}$, d'où, substituant les valeurs de ces deux derniers termes : $\frac{A\Theta}{\Theta\Gamma} < \frac{3013\frac{3}{4} + 2911}{780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{5924\frac{3}{4}}{780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{\frac{4}{13} \times 5924\frac{3}{4}}{\frac{4}{13} \times 780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{1823}{240}$. D'autre part, $\frac{\overline{A\Theta}^2}{\overline{\Theta\Gamma}^2} < \frac{1823^2}{240^2}$, d'où $\frac{\overline{A\Theta}^2 + \overline{\Theta\Gamma}^2}{\overline{\Theta\Gamma}^2} < \frac{1823^2 + 240^2}{240^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{\Theta\Gamma}^2} < \frac{3380929}{57600}$, d'où, comme le texte : $\frac{A\Gamma}{\Theta\Gamma} < \frac{1838\frac{9}{11}}{240}$.

2. On aura de même : $\frac{AK}{K\Gamma} = \frac{A\Gamma + A\Theta}{\Theta\Gamma} = \frac{A\Gamma}{\Theta\Gamma} + \frac{A\Theta}{\Theta\Gamma}$. et, par substitution des valeurs trouvées pour ces deux derniers termes, il vient : $\frac{AK}{K\Gamma} < \frac{1838\frac{9}{11} + 1823}{240}$, ou $\frac{AK}{K\Gamma} < \frac{3661\frac{9}{11}}{240}$, ou $\frac{AK}{K\Gamma} < \frac{\frac{11}{40} 3661\frac{9}{11}}{\frac{11}{40} 240}$, ou $\frac{AK}{K\Gamma} < \frac{1007}{66}$. D'autre part, $\frac{\overline{AK}^2}{\overline{K\Gamma}^2} < \frac{1007^2}{66^2}$, d'où : $\frac{\overline{AK}^2 + \overline{K\Gamma}^2}{\overline{K\Gamma}^2} < \frac{1007^2 + 66^2}{66^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{K\Gamma}^2} < \frac{1018405}{14356}$, d'où, comme le texte : $\frac{A\Gamma}{K\Gamma} < \frac{1009\frac{1}{6}}{66}$.

3. On aura de même: $\frac{A\Lambda}{\Lambda\Gamma} = \frac{A\Gamma + AK}{K\Gamma} = \frac{A\Gamma}{K\Gamma} + \frac{AK}{K\Gamma}$, et, par substitution des valeurs

précédentes : $\frac{A\Lambda}{\Lambda\Gamma} < \frac{1009\frac{1}{6}}{66} + \frac{1007}{66}$, ou $\frac{A\Lambda}{\Lambda\Gamma} < \frac{2016\frac{1}{6}}{66}$. D'autre part,

$\frac{\overline{A\Lambda}^2 + \overline{\Lambda\Gamma}^2}{\overline{\Lambda\Gamma}^2} < \frac{(2016\frac{1}{6})^2 + 66^2}{66^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{\Lambda\Gamma}^2} < \frac{4069284\frac{1}{36}}{4356}$, d'où, comme le texte: $\frac{A\Gamma}{\Lambda\Gamma} < \frac{2017\frac{1}{4}}{66}$.

1. Sous-entendu : περίμετρος, le périmètre (du cercle).

2. La relation de la note avant-précédente donne, par inversion : $\frac{\Lambda\Gamma}{A\Gamma} > \frac{66}{2017\frac{1}{4}}$, d'où, observant que $96 \times \Lambda\Gamma$ = périmètre polygone inscrit de 96 côtés :
$\frac{\text{périmètre polygone de 96 côtés}}{\text{diamètre cercle}} > \frac{96 \times 66}{2017\frac{1}{4}}$, ou $> \frac{6336}{2017\frac{1}{4}}$. Or, $\frac{6336}{2017\frac{1}{4}} > 3\frac{10}{71}$, d'où périmètre polygone de 96 côtés $> 3\frac{10}{71}$ diamètre cercle, d'où, à fortiori, suivant le texte : Circonférence cercle $> 3\frac{10}{71}$ diamètre.

Archimedes' Proposition 2 (improving the Rhind papyrus):

$$\text{Area of circle} = d^2 \cdot \frac{11}{14} \qquad (d = \text{ diameter})$$

Proof by Prop. 1 and Prop. 3.

... and what happended the next 1800 years ?

Zu Chongzhi (China 480 A.D.) computed with same method:

$$d \cdot 3.1415926 < \text{Perimeter of circle} < d \cdot 3.1415927$$

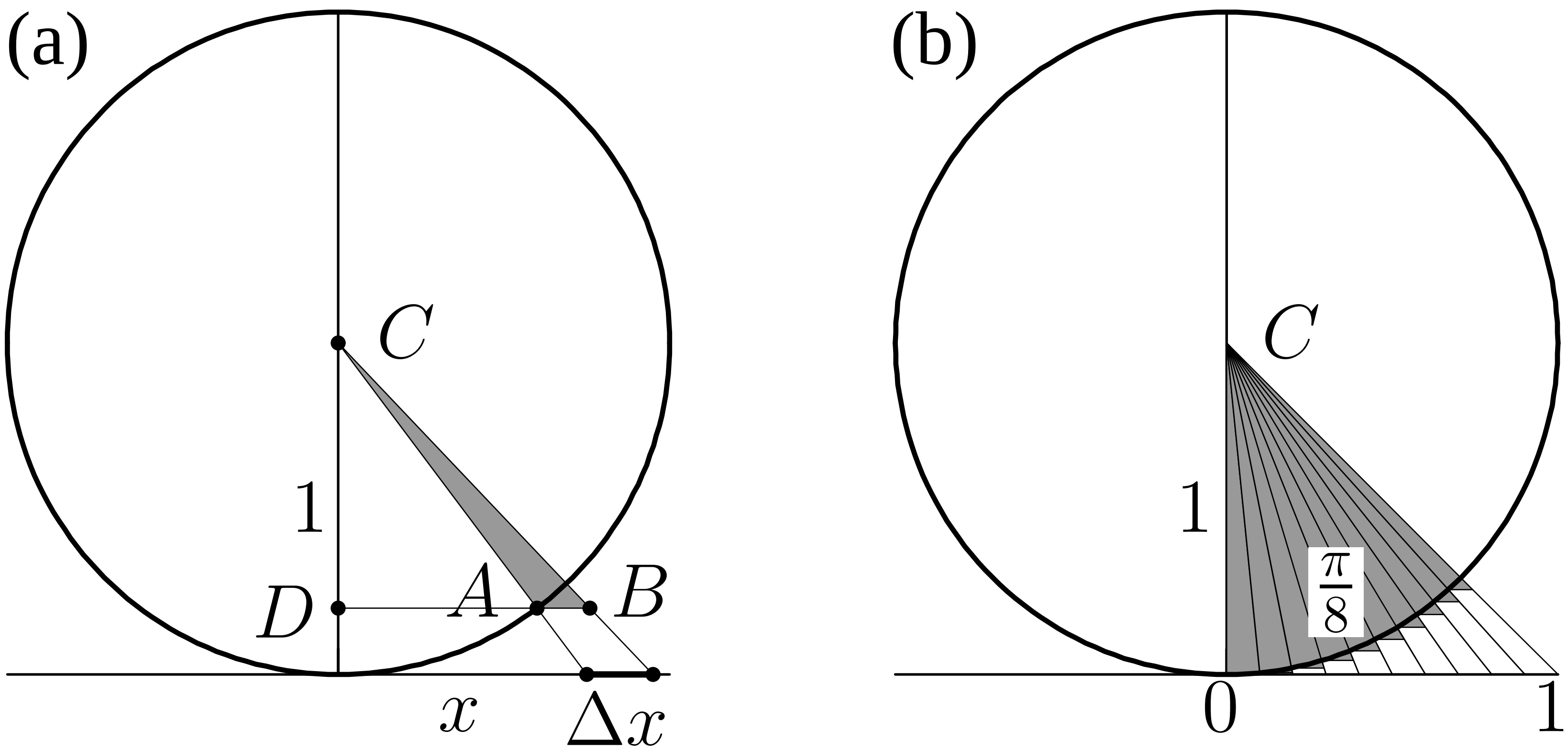
Adriaan van Roomen (1593 A.D.) with same method:

$$\text{Perimeter of circle} = d \cdot 3.1415926535897932$$

Ludolph van Ceulen (1596) with same method:

$$\text{Per. of circle} = d \cdot 3.14159265358979323846264338327950288$$

New Progress: Leibniz (Invention of Calculus, ≈ 1675 A.D.)



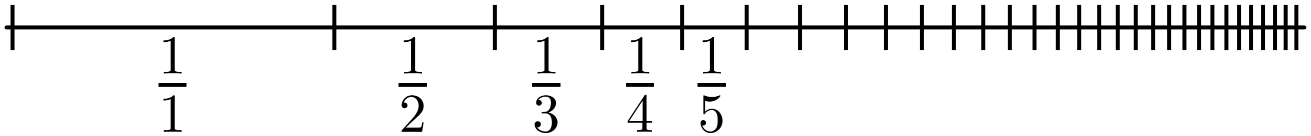
Pythagoras and Thales: $CD = \frac{1}{\sqrt{1+x^2}}$ and $AB = \frac{\Delta x}{\sqrt{1+x^2}}$. Hence

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2} = \int_0^1 (1 - x^2 + x^4 - x^6 + \ldots)\, dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

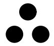
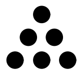
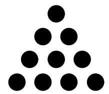
New Challenges: (Jakob Bernoulli 1689)

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\ldots = \textbf{?} \quad \text{or} \quad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots = \textbf{?}$$

Answer: both diverge.

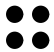
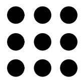
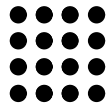


Triangular Numbers:

$\bullet$ $= 1$,  $= 3$,  $= 6$,  $= 10$, $\ldots$

Theorem:

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\ldots = 2\,.$$

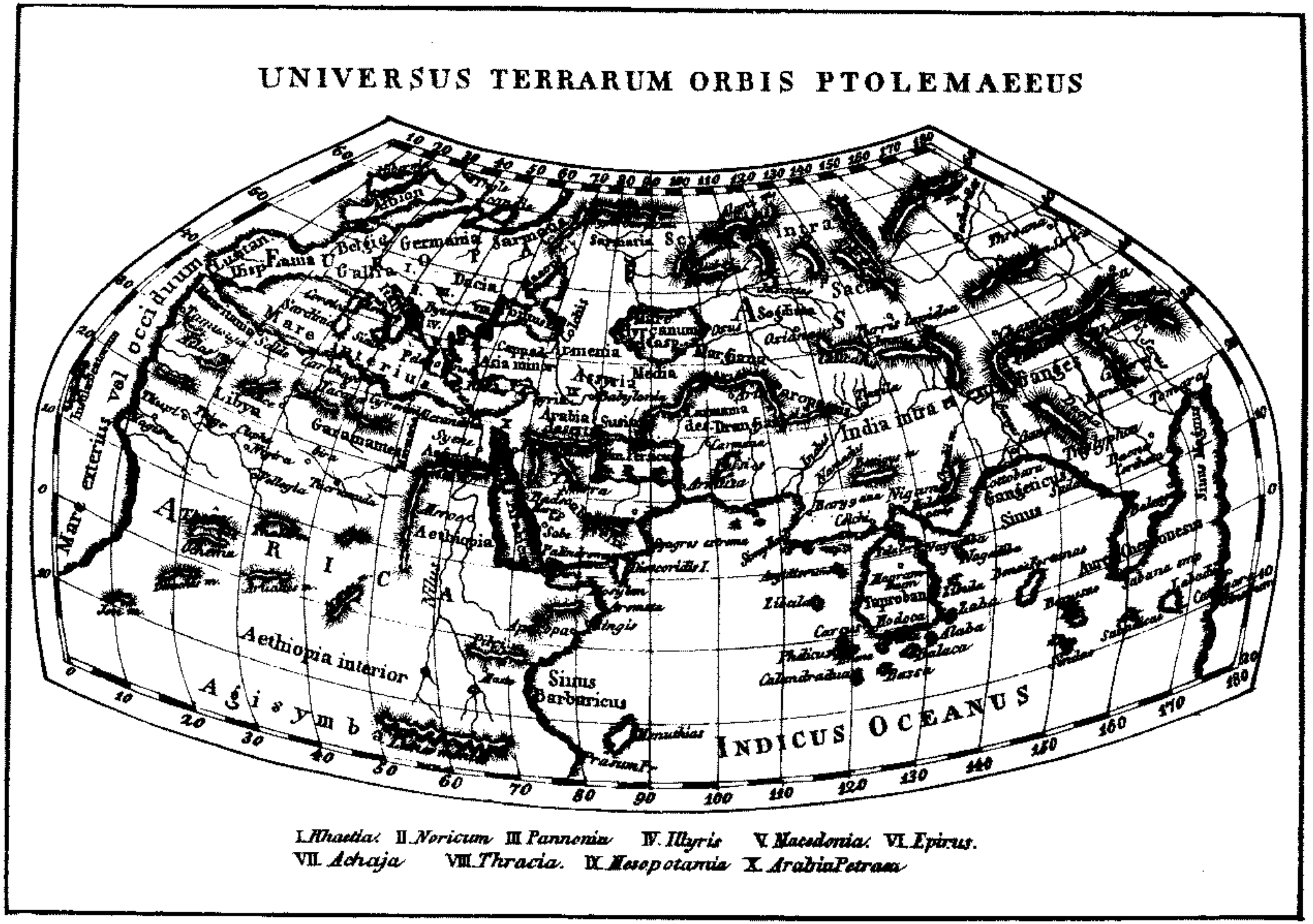
Great Challenge: $\bullet$ $= 1$,  $= 4$,  $= 9$,  $= 16$, $\ldots$

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots = \textbf{?}$$

Here we stop, wait for the arrival of Euler, and return to another Giant of Greek science ...

II. Measuring Angles. Cl. Ptolemaios (150 A.D.)

Measuring the Universe, Planets, Earth (Γῆ … μετρέω):

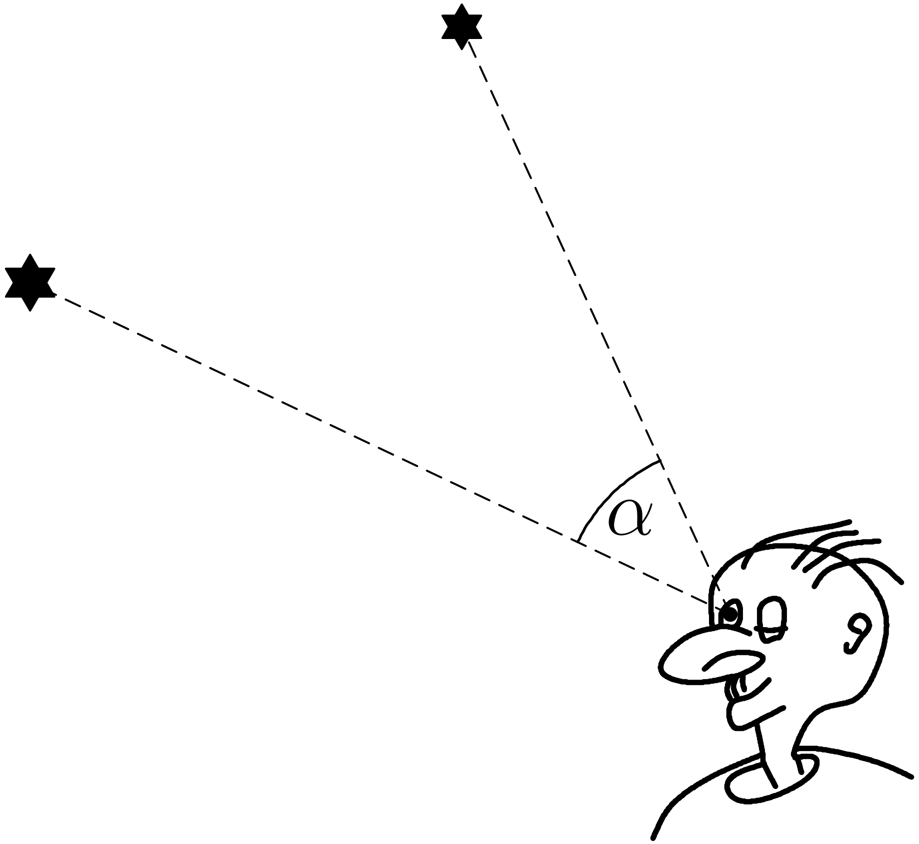


Ptolemy measured coordinates of 8000 cities, among ...

ὑφ᾽ οὓς Παρίσιοι, καὶ πόλις
Παρισίων Λουκοτεκία .... κγ ∠ μη ∠

II. Measuring Angles.

Problem. Measure angle α !!



II. Measuring Angles.

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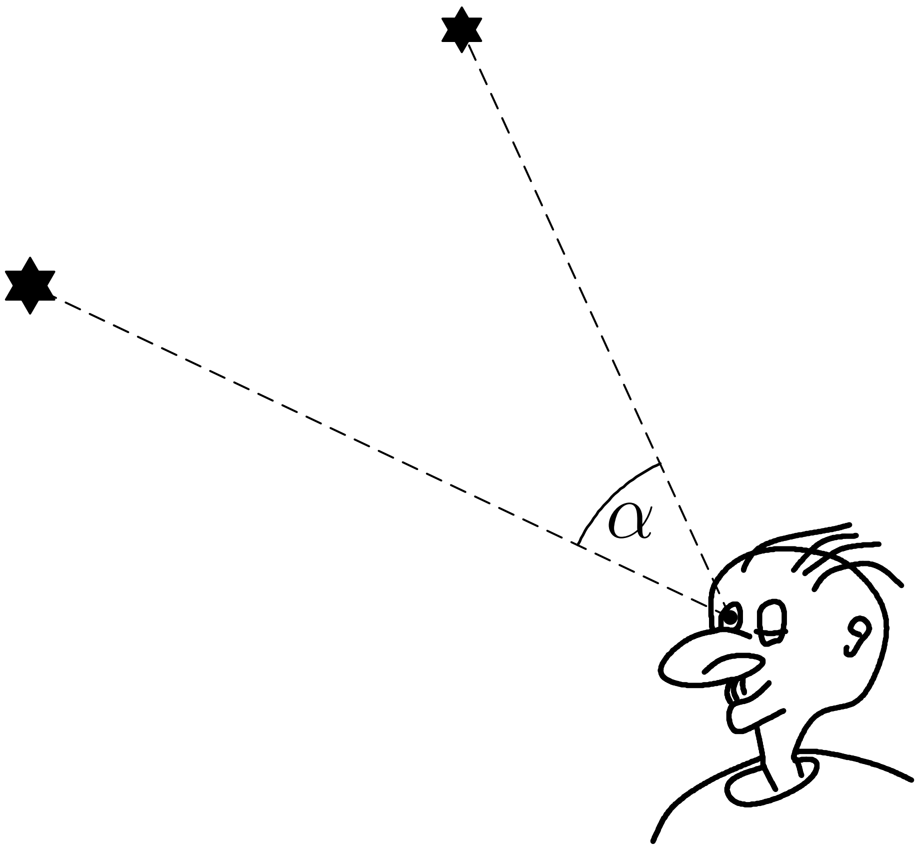
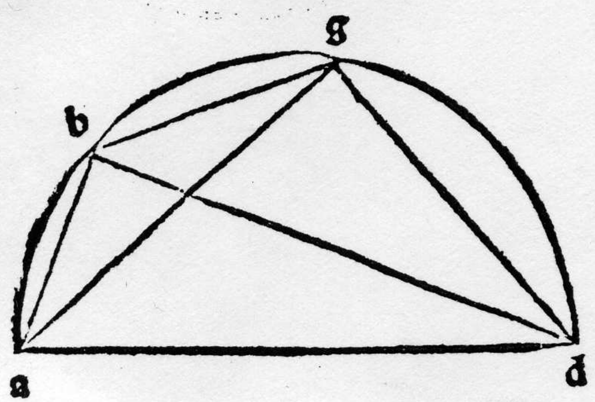


Solution. Put sticks of length 1 (or 60), measure distance c.

The chord function. Cl. Ptolemaios (150 A.D., printed 1496)



Propositio iiij.

Notis chordis inequalium arcuum in semicirculo: arcus quo maior minorē superat chorda nota fiet.
¶ Ut in semicirculo.a.b.d.supra diametrū.a.d.note sint chorde.a.b.a.g. Dico notam fieri chordam.b.g.nam per correlarium prime huius note etiam fient chorde.b.d.⁊.g.d. ¶ Sint in quadrilatero.a.b.g.d.diametri.a.g.⁊.b.d.note.sunt ⁊ latera.a.b.⁊.g.d.opposita nota.igit̄ per premissam quod fit ex.a.d.in.b.g.notū iet.Sed.a.d.est nota:quia diameter circuli.ideo.b.g.nota fiet: q̄ querebat̄. Per hāc plurimorꝫ arcuū chordas cognosces. Repies enī chordā arcus quo q̄nta pars circūferentie sextā supat.s.chordā arcus.12.graduū:⁊ sic de alijs.

Ptolemaios computed chord of $\alpha \pm \beta$ using Eucl. III.20 and "Ptolemy's Lemma".

Ptolemy's Table of the Chords. (150 A.D., printed 1813)

TABLE DES DROITES INSCRITES DANS LE CERCLE.

ARCS.		CORDES.			TRENTIÈMES DES DIFFÉRENCES.			
Degrés	Min.	Part. du Diam.	Prim	Secon.	Part.	Prim.	Secon.	Tierc.
0	30	0	31	25	0	1	2	50
1	0	1	2	50	0	1	2	50
1	30	1	34	15	0	1	2	50
2	0	2	5	40	0	1	2	50
2	30	2	37	4	0	1	2	48
3	0	3	8	28	0	1	2	48
3	30	3	39	52	0	1	2	48
4	0	4	11	16	0	1	2	47
4	30	4	42	40	0	1	2	47
5	0	5	14	4	0	1	2	46
5	30	5	45	27	0	1	2	45
6	0	6	16	49	0	1	2	44
6	30	6	48	11	0	1	2	43
7	0	7	19	33	0	1	2	42
7	30	7	50	54	0	1	2	41
8	0	8	22	15	0	1	2	40
8	30	8	53	35	0	1	2	39
9	0	9	24	54	0	1	2	38

ΚΑΝΟΝΙΟΝ ΤΩΝ ΕΝ ΚΥΚΛΩ ΕΥΘΕΙΩΝ.

ΠΕΡΙΦΕΡΕΙΩΝ.		ΕΥΘΕΙΩΝ.			ΕΞΗΚΟΣΤΩΝ.			
Μοιρῶν.		Μ.	Π.	Δ.	Μ.	Π.	Δ.	Τ.
ō	ς''	ō	λα	κε	ō	α	β	ν
α	ō	α	β	ν	ō	α	β	ν
α	ς''	α	λδ	ιε	ō	α	β	ν
β	ō	β	ε	μ	ō	α	β	ν
β	ς''	β	λζ	δ	ō	α	β	μη
γ	ō	γ	η	κη	ō	α	β	μη
γ	ς''	γ	λϑ	νβ	ō	α	β	μη
δ	ō	δ	ια	ις	ō	α	β	μζ
δ	ς''	δ	μβ	μ	ō	α	β	μζ
ε	ō	ε	ιδ	δ	ō	α	β	μς
ε	ς''	ε	με	κζ	ō	α	β	με
ς	ō	ς	ις	μϑ	ō	α	β	μδ
ς	ς''	ς	μη	ια	ō	α	β	μγ
ζ	ō	ζ	ιϑ	λγ	ō	α	β	μβ
ζ	ς''	ζ	ν	νδ	ō	α	β	μα
η	ō	η	κβ	ιε	ō	α	β	μ
η	ς''	η	νγ	λε	ō	α	β	λϑ
ϑ	ō	ϑ	κδ	νδ	ō	α	β	λη

Another Part of Ptolemy's Table.

πα	ϛ″	οη	ιϑ	νβ	ō	ō	μζ	λα
πβ	ō	οη	μγ	λη	ō	ō	μζ	κ
πβ	ϛ″	οϑ	ζ	ιη	ō	ō	μζ	ϑ
πγ	ō	οϑ	λ	νβ	ō	ō	μϛ	νη
πγ	ϛ″	οϑ	νδ	κα	ō	ō	μϛ	μζ
πδ	ō	π	ιζ	με	ō	ō	μϛ	λϛ
πδ	ϛ″	π	μα	γ	ō	ō	μϛ	κε
πε	ō	π	δ	ιε	ō	ō	μϛ	ιδ
πε	ϛ″	πα	κζ	κβ	ō	ō	μϛ	γ
πϛ	ō	πα	ν	κδ	ō	ō	με	νβ
πϛ	ϛ″	πβ	ιγ	ιϑ	ō	ō	με	μ
πζ	ō	πβ	λϛ	ϑ	ō	ō	με	κϑ
πζ	ϛ″	πβ	νη	νδ	ο	ō	με	ιη
πη	ō	πγ	κα	λγ	ō	ō	με	ϛ
πη	ϛ″	πγ	μδ	ϛ	ō	ō	μδ	νε
πϑ	ō	πδ	ϛ	λγ	ō	ō	μδ	μγ
πϑ	ϛ″	πδ	κη	νε	ō	ō	μδ	λα
ϟ	ο	πδ	να	ι	ō	ō	μδ	κ

81	30	78	19	52	0	0	47	31
82	0	78	43	38	0	0	47	20
82	30	79	7	18	0	0	47	9
83	0	79	30	52	0	0	46	58
83	30	79	54	21	0	0	46	47
84	0	80	17	45	0	0	46	36
84	30	80	41	3	0	0	46	25
85	0	80	4	15	0	0	46	14
85	30	81	27	22	0	0	46	3
86	0	81	50	24	0	0	45	52
86	30	82	13	19	0	0	45	40
87	0	82	36	9	0	0	45	29
87	30	82	58	54	0	0	45	18
88	0	83	21	33	0	0	45	6
88	30	83	44	6	0	0	44	55
89	0	84	6	33	0	0	44	43
89	30	84	28	55	0	0	44	31
90	0	84	51	10	0	0	44	20

Control of Ptolemy's Table.

TABLE DES DROITES INSCRITES DANS LE CERCLE.

ARCS.		CORDES.			TRENTIÈMES DES DIFFÉRENCES.			
Degrés	Min.	Part. du Diam.	Prim	Secon.	Part.	Prim.	Secon.	Tierc.
0	30	0	31	25	0	1	2	50
1	0	1	2	50	0	1	2	50
1	30	1	34	15	0	1	2	50
2	0	2	5	40	0	1	2	50
2	30	2	37	4	0	1	2	48
3	0	3	8	28	0	1	2	48
3	30	3	39	52	0	1	2	48
4	0	4	11	16	0	1	2	47
4	30	4	42	40	0	1	2	47
5	0	5	14	4	0	1	2	46
5	30	5	45	27	0	1	2	45
6	0	6	16	49	0	1	2	44
6	30	6	48	11	0	1	2	43
7	0	7	19	33	0	1	2	42
7	30	7	50	54	0	1	2	41
8	0	8	22	15	0	1	2	40
8	30	8	53	35	0	1	2	39
9	0	9	24	54	0	1	2	38

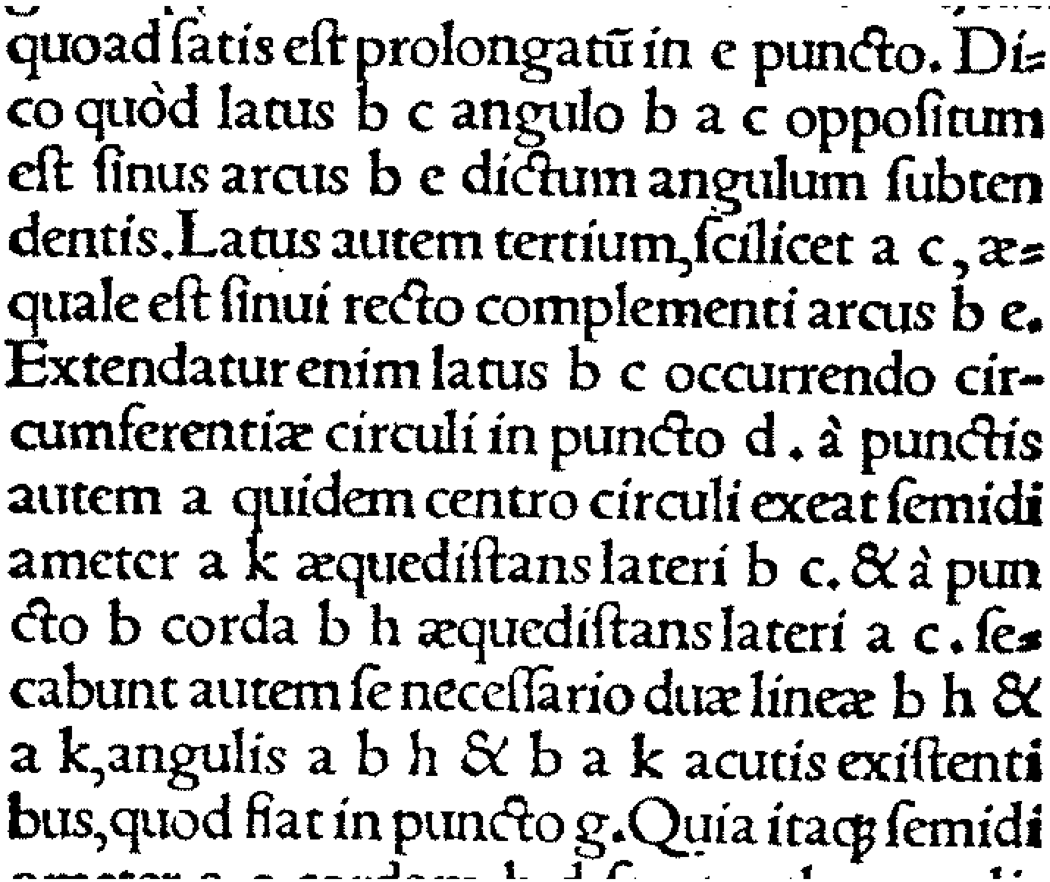
0	30	0	31	25
1	0	1	2	50
1	30	1	34	15
2	0	2	5	39
2	30	2	37	4
3	0	3	8	28
3	30	3	39	53
4	0	4	11	17
4	30	4	42	40
5	0	5	14	4
5	30	5	45	27
6	0	6	16	49
6	30	6	48	11
7	0	7	19	33
7	30	7	50	54
8	0	8	22	15
8	30	8	53	35
9	0	9	24	54

Control.

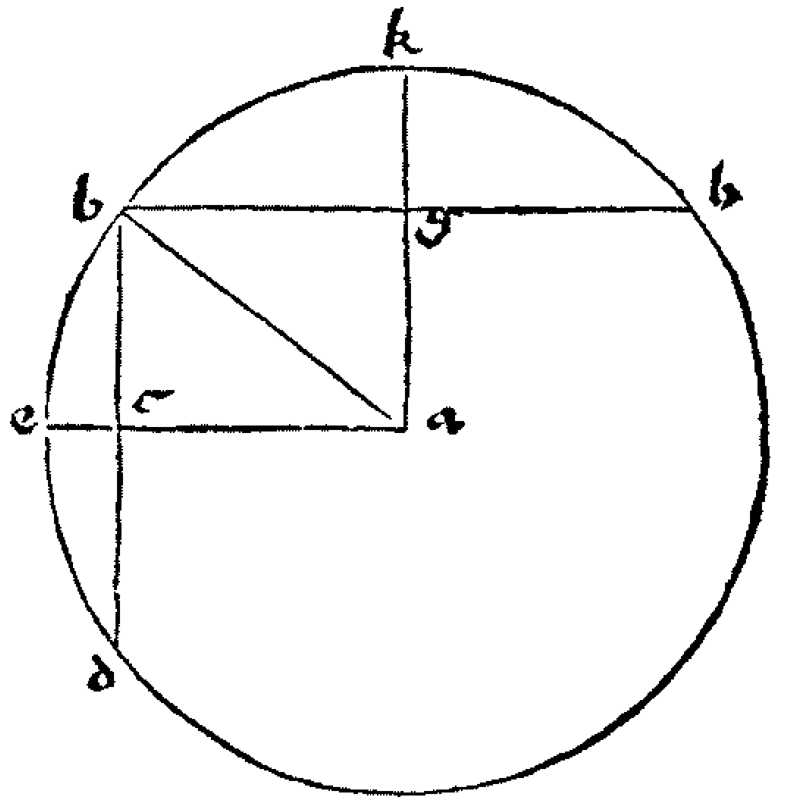
81	30	78	19	52	0	0	47	31
82	0	78	43	38	0	0	47	20
82	30	79	7	18	0	0	47	9
83	0	79	30	52	0	0	46	58
83	30	79	54	21	0	0	46	47
84	0	80	17	45	0	0	46	36
84	30	80	41	3	0	0	46	25
85	0	80	4	15	0	0	46	14
85	30	81	27	22	0	0	46	3
86	0	81	50	24	0	0	45	52
86	30	82	13	19	0	0	45	40
87	0	82	36	9	0	0	45	29
87	30	82	58	54	0	0	45	18
88	0	83	21	33	0	0	45	6
88	30	83	44	6	0	0	44	55
89	0	84	6	33	0	0	44	43
89	30	84	28	55	0	0	44	31
90	0	84	51	10	0	0	44	20

81	30	78	19	52
82	0	78	43	38
82	30	79	7	17
83	0	79	30	52
83	30	79	54	21
84	0	80	17	44
84	30	80	41	2
85	0	81	4	15
85	30	81	27	22
86	0	81	50	23
86	30	82	13	19
87	0	82	36	9
87	30	82	58	54
88	0	83	21	32
88	30	83	44	5
89	0	84	6	33
89	30	84	28	54
90	0	84	51	10

The Sine and Cosines (“sinus recto complementari”), Trigonometric Circle. Regiomontanus (publ. 1533 A.D.)



quoad satis est prolongatũ in e puncto. Di=
co quòd latus b c angulo b a c oppositum
est sinus arcus b e dictum angulum subten
dentis. Latus autem tertium, scilicet a c, æ=
quale est sinui recto complementi arcus b e.
Extendatur enim latus b c occurrendo cir-
cumferentiæ circuli in puncto d. à punctis
autem a quidem centro circuli exeat semidi
ameter a k æquedistans lateri b c. & à pun
cto b corda b h æquedistans lateri a c. se=
cabunt autem se necessario duæ lineæ b h &
a k, angulis a b h & b a k acutis existenti
bus, quod fiat in puncto g. Quia itaq; semidi

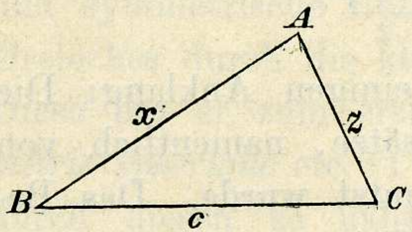


Around 1464, Regiomontanus computed a table (“SEQVITVR NVNC EIVSDEM IOANNIS Regiomontani tabula sinuum, per singula minuta extensa . . .”) giving the sine of all angles at intervals of 1 minute, with five decimals.

Tentatives to turn Latin text into Formulas:

$$\frac{(:BC) \div (AB + AC) \text{ in } \square\, R}{\lvert AB, \text{ in } AC \rvert} = (:A),$$
$$\text{— Sin —}$$

J.J. Stampioen (1632)



AB mit x, AC mit z, BC mit c und schrieb:

$r \cdot\cdot \text{S.2. (anguli } BAC) :: 2xz \cdot\cdot xx + zz - cc$

Durch die ungleichmäſsige Bezeichnungsweise der Funktionen wurden die Rechnungen natürlich sehr unübersichtlich, und das Schluſsresultat muſste

J. Kresa (St. Pet., 1720)

cosinum anguli ad B fore $= \dfrac{rq - Cc}{Ss} r$.

sinu cruris $AB = S$, cosinus eiusdem $= C$,
sinu cruris $BC = s$ et cosinu $= c$,
cosinu baseos $AC = q$, et radio $= r$;

F.C. Maier (St. Pet., 1727)

cosinum anguli ad B fore $= \dfrac{rq - Cc}{Ss} r$. sinu cruris $AB = S$, cosinus eiusdem $= C$, sinu cruris $BC = s$ et cosinu $= c$, cosinu baseos $AC = q$, et radio $= r$;	F.C. Maier (St. Pet., 1727)

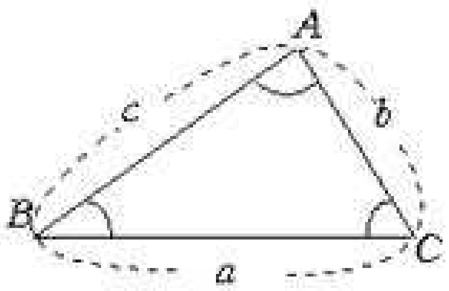
cosinum anguli ad B fore $= \frac{rq - Cc}{Ss} r$. sinu cruris $AB = S$, cosinus eiusdem $= C$, sinu cruris $BC = s$ et cosinu $= c$, cosinu baseos $AC = q$, et radio $= r$;	F.C. Maier (St. Pet., 1727)
$\cos:$anguli $A = \frac{\cos:BC - \cos:AB \cdot \cos:AC}{s\,AB \cdot s\,AC}$, posito radio vel sinu toto 1.	L. Euler (E14, 1729)

cosinum anguli ad B fore $= \dfrac{rq - Cc}{Ss} r$. sinu cruris $AB = S$, cosinus eiusdem $= C$, sinu cruris $BC = s$ et cosinu $= c$, cosinu baseos $AC = q$, et radio $= r$;	F.C. Maier (St. Pet., 1727)
$\cos : \text{anguli } A = \dfrac{\cos : BC - \cos : AB \cdot \cos : AC}{s\,AB \cdot s\,AC}$, posito radio vel sinu toto 1.	L. Euler (E14, 1729)
$\cos A = \dfrac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$	L. Euler (E214, 1753)
$\cos a = \cos b \cos c + \sin b \sin c \cos A$	Handbook (Washing., 1965)

. . . and another half of a century later . . .

△ABC의 변 a, b, c와 각 A, B, C의 코사인 사이에는 다음과 같은 코사인법칙이 성립한다.

코사인법칙



제일코사인법칙	제이코사인법칙
$a = b\cos C + c\cos B$	$a^2 = b^2 + c^2 - 2bc\cos A$
$b = c\cos A + a\cos C$	$b^2 = c^2 + a^2 - 2ca\cos B$
$c = a\cos B + b\cos A$	$c^2 = a^2 + b^2 - 2ab\cos C$

“I don’t need to explain further the simplicity and elegance introduced by the trigonometric formulas and theorems. Although it is such a simple idea, it required 2000 years to be discovered.”

(A. v. Braunmühl, Bibl. Math., vol. 1, 1900, p. 73)

“Similarly to Johann Bernoulli, who has turned the logarithms into analytic functions, I believe to have been the first to introduce the Sinus and the Tangents of angles into the Calculus, so that they could be treated as any other quantity with all operations without obstacles. Even if this seems not to be of great importance, it can be said that this notation has given the analysis so important tools, that nearly a new field of research has been created, which the geometers have since then developped with so much success ...”

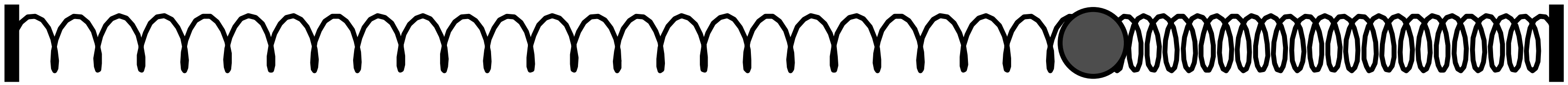
(Euler E246, 1754, orig. in Latin)

“Nearly a new field ...”:

- Theory of Sound Waves (Lagrange 1759)
- Secular Perturbations of Planets (Lagrange 1770, Leverrier 1846, Einstein 1916)
- Fourier series and Theory of Heat (Fourier 1807, 1822)
- Electromagnetic Waves (Light, Radio, ... Maxwell 1865)
- Schrödinger Wave Equation (Schrödinger 1926)
- Treatment of Sound, Images (FFT, JPEG, MPEG, MP3 ,...)

III. Mechanics (Dynamics)).

Problem. How moves a mass point attached to springs ?



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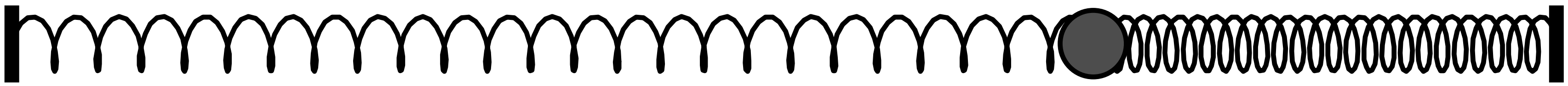
Problem. How moves a mass point attached to springs ?



R. Hooke (1678): Force $f = -K \cdot y$.

I. Newton (1687): "Change of movement is proportional to the acting force" (orig. in Latin); in formulas:

$$m \cdot \dot{v} = f \qquad \dot{y} = v \ .$$

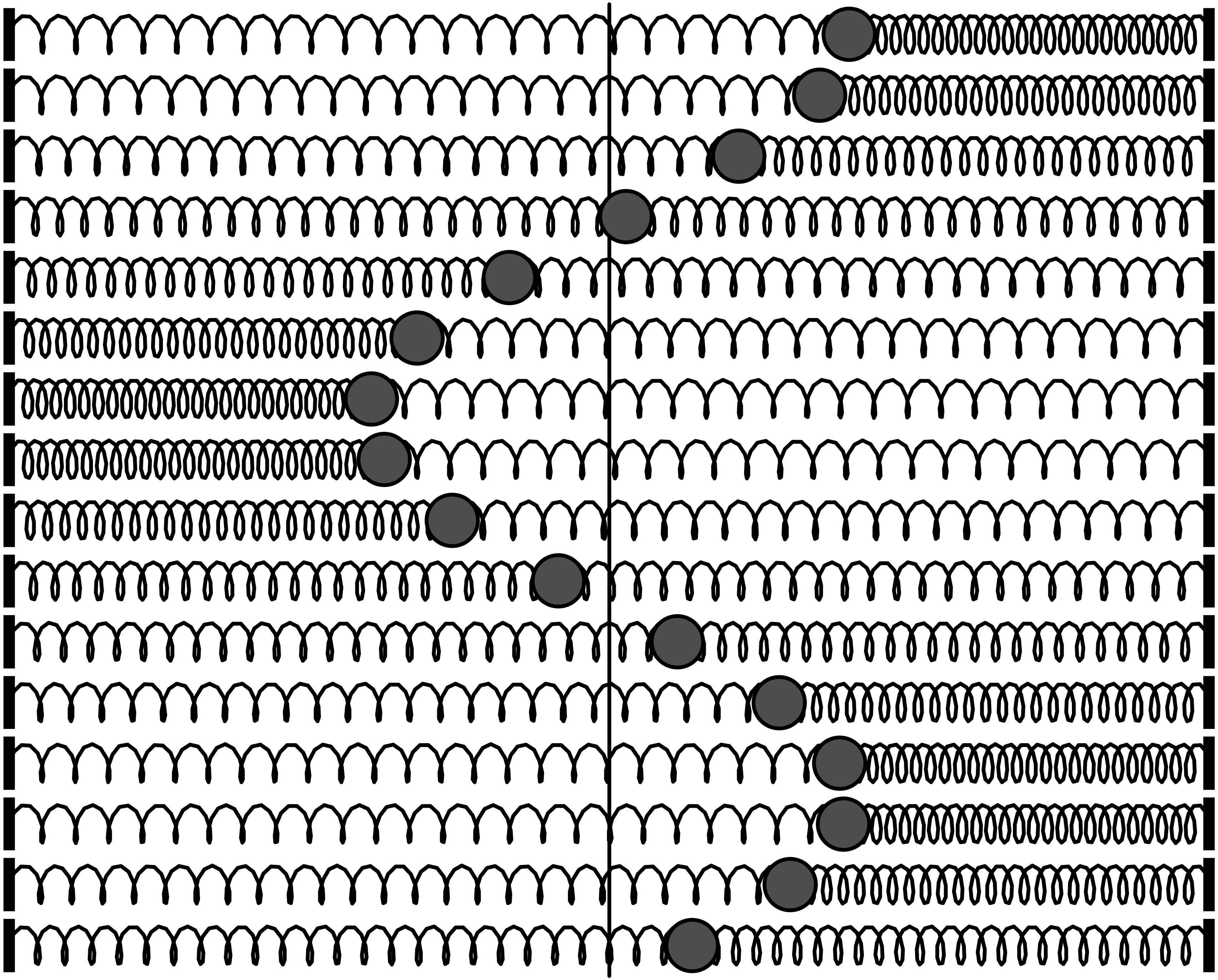
L. Euler (E15, 1736): Have to solve differential equation

$$m \cdot \ddot{y} + K \cdot y = 0 \qquad \text{or} \qquad \frac{ddy}{dt^2} + k^2 y = 0 \ .$$

Solutions.

$$y = \gamma e^{+ikt} + \delta e^{-ikt} \quad \text{or} \quad y = \alpha \cdot \sin(kt + \beta) \ .$$

Solution.

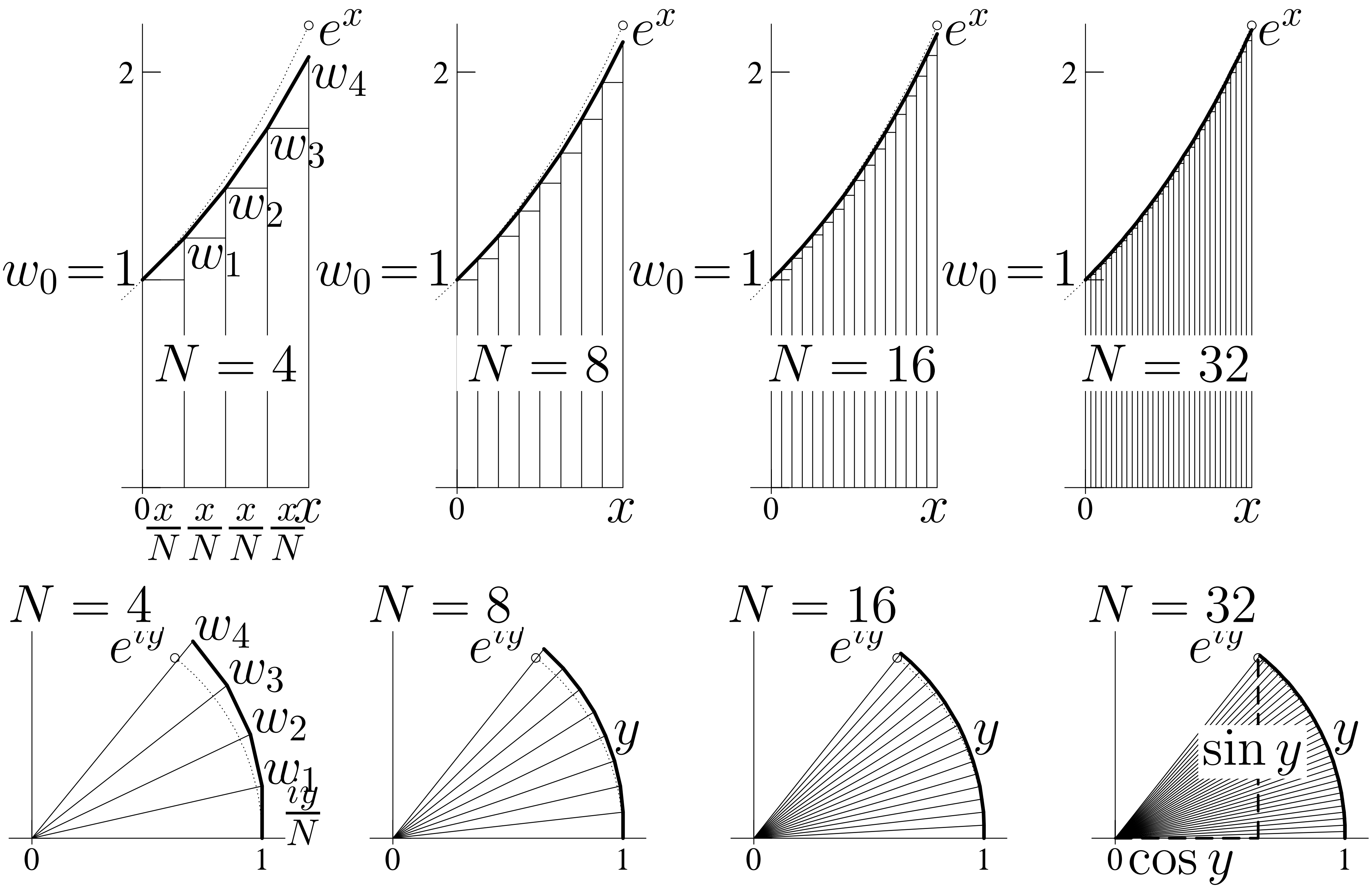


“autem exponentialibus in series conversis”

use $i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ \ldots$:

$$
\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \ldots \\
&= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \ldots\right)}_{\cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \ldots\right)}_{\sin x} \\
&= \cos x + i \sin x.
\end{aligned}
$$

Geometric Proof: $e^x = (1 + \frac{x}{N})^N, \qquad e^{iy} = (1 + \frac{iy}{N})^N.$

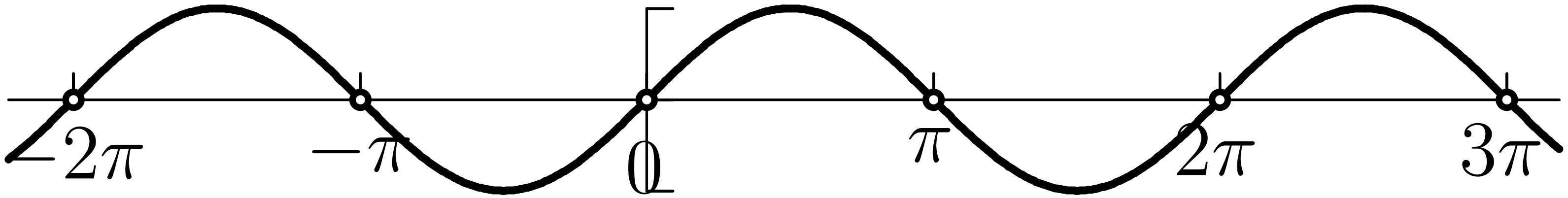


“Basel” Problem: Find value of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = \; ?$$

Euler’s audacious idea **E41:** (1735):
look at the series for $\sin s$ as if it were a polynomial:

$$\sin s = s - \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \ldots$$



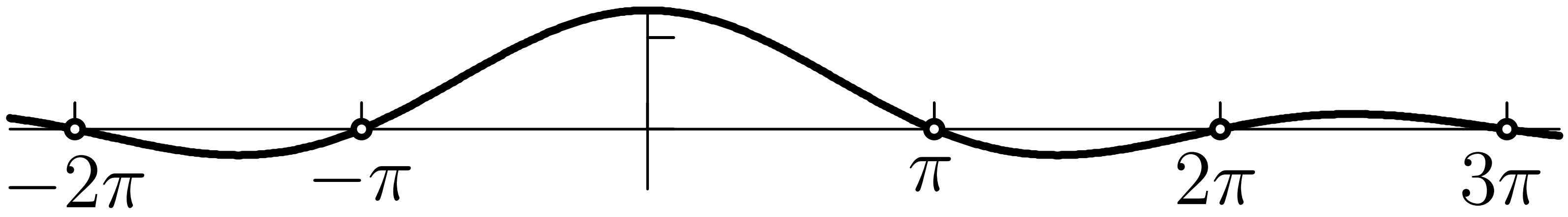
Roots: $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

“Basel” Problem: cont’d.

$$s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dots$$

divide by s:

$$1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dots$$



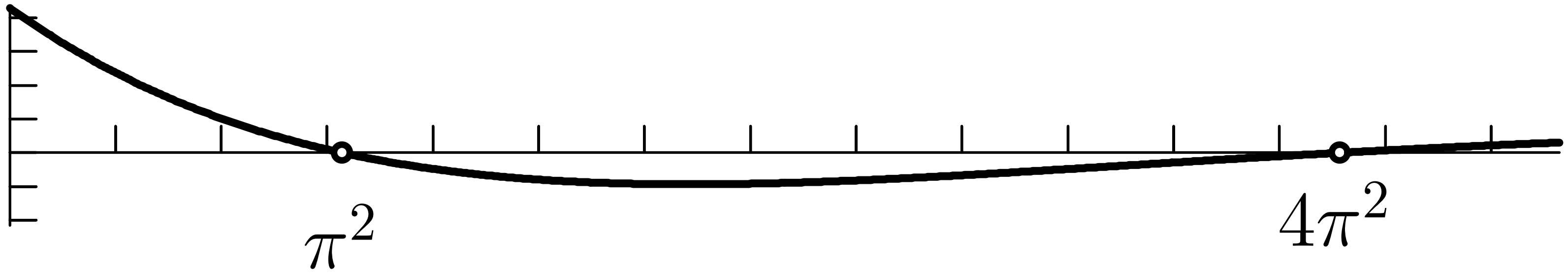
Roots: $\pm\,\pi, \pm 2\pi, \pm 3\pi, \dots$

“Basel” Problem: cont’d.

$$1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots$$

replace s^2 by t:

$$1 - \frac{t}{1 \cdot 2 \cdot 3} + \frac{t^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots$$



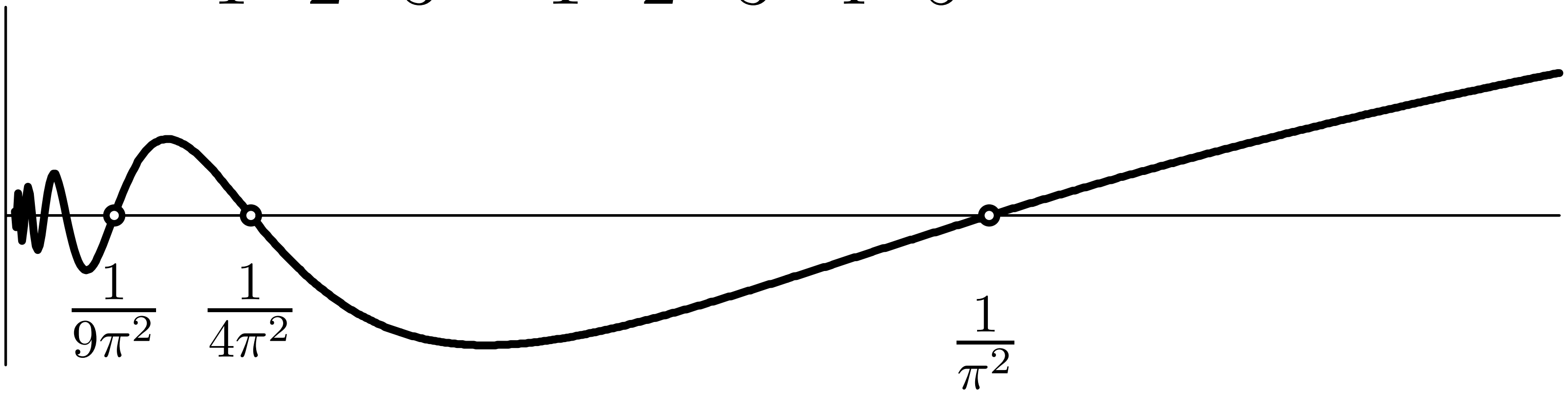
Roots: $\pi^2,\ 4\pi^2,\ 9\pi^2, \ldots$

“Basel” Problem: cont’d.

$$1 - \frac{t}{1 \cdot 2 \cdot 3} + \frac{t^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots$$

replace t by $\frac{1}{x}$, multiply by x^N:

$$x^N - \frac{x^{N-1}}{1 \cdot 2 \cdot 3} + \frac{x^{N-2}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots$$



Roots: $\frac{1}{\pi^2}, \frac{1}{4\pi^2}, \frac{1}{9\pi^2}, \ldots$

$$x^N - \frac{x^{N-1}}{1 \cdot 2 \cdot 3} + \frac{x^{N-2}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots$$

Back to first year algebra: The polynomial

$$p = x^N - \alpha x^{N-1} + \beta x^{N-2} - \gamma x^{N-3} + \ldots$$

$$\Rightarrow \quad \alpha = x_1 + x_2 + x_3 + \ldots \qquad \text{(Viète)}$$

The break-through E41: (1735)

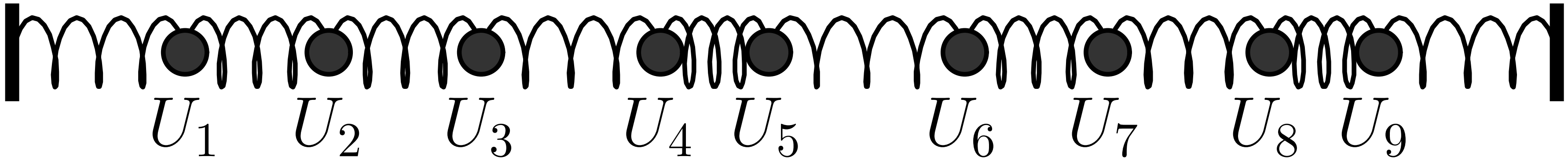
"One of Euler's most sensational early discoveries, perhaps the one which established his growing reputation most firmly, was his summation of the series $\sum_1^\infty n^{-2}$ (...). This was a famous problem, first formulated by P. Mengoli in 1650 ; it had resisted the efforts of all earlier analysts, including Leibniz and the Bernoullis." (A. Weil, *Number theory*, 1984, p. 184)

$$x^N - \frac{x^{N-1}}{1\cdot 2\cdot 3} + \frac{x^{N-2}}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \dots$$

has "roots" $\frac{1}{\pi^2}, \frac{1}{4\pi^2}, \frac{1}{9\pi^2}, \dots$. Hence, by Viète,

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots = \frac{1}{6} \text{ or } 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \dots = \frac{\pi^2}{6}.$$

Back to Mechanics: Theory of Sound (Lagrange 1759)



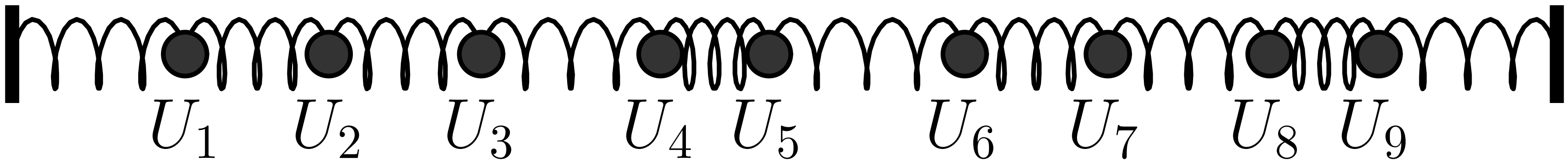
Principle of Mechanics:

$$\frac{d^2U_i}{dt^2} = K\cdot((U_{i-1}-U_i)+(U_{i+1}-U_i)) = K\cdot(U_{i-1}-2U_i+U_{i+1})$$

When number of particles $\rightarrow \infty$:

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2}$$ (famous wave equation).

Back to Mechanics: Theory of Sound (Lagrange 1759)



Principle of Mechanics:

$$\frac{d^2U_i}{dt^2} = K\cdot((U_{i-1}-U_i)+(U_{i+1}-U_i)) = K\cdot(U_{i-1}-2U_i+U_{i+1})$$

When number of particles $\to\infty$:

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2}$$

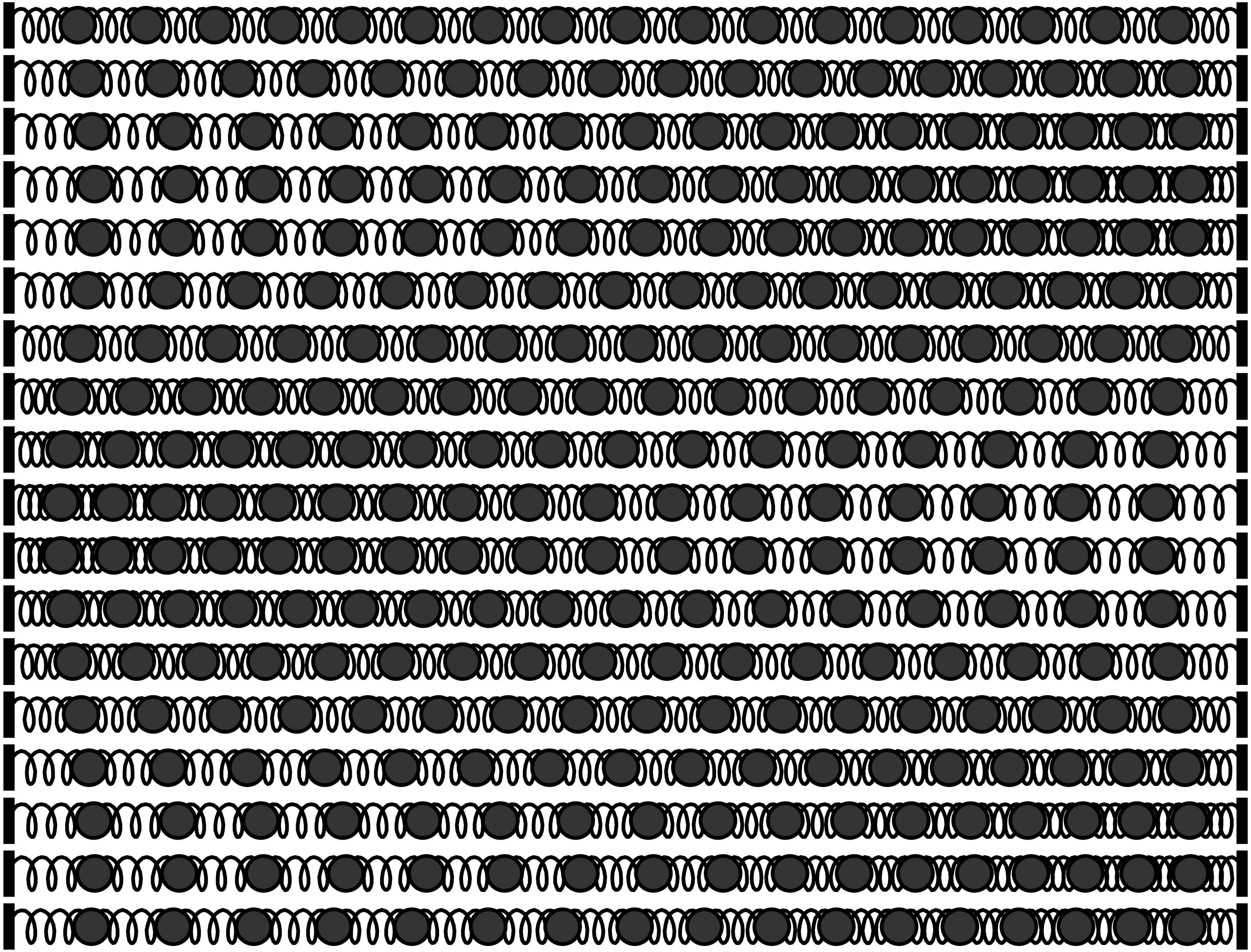
(famous wave equation).

Solution (remember: $y''+k^2y=0 \quad\Rightarrow\quad y=\sin(kx)$):

$$\underbrace{\frac{\partial^2 u}{\partial t^2}}_{-a^2k^2u} = a^2\;\underbrace{\frac{\partial^2 u}{\partial x^2}}_{-k^2u} \qquad\Rightarrow\qquad u=\sin(kx)\cdot\sin(akt)\,.$$

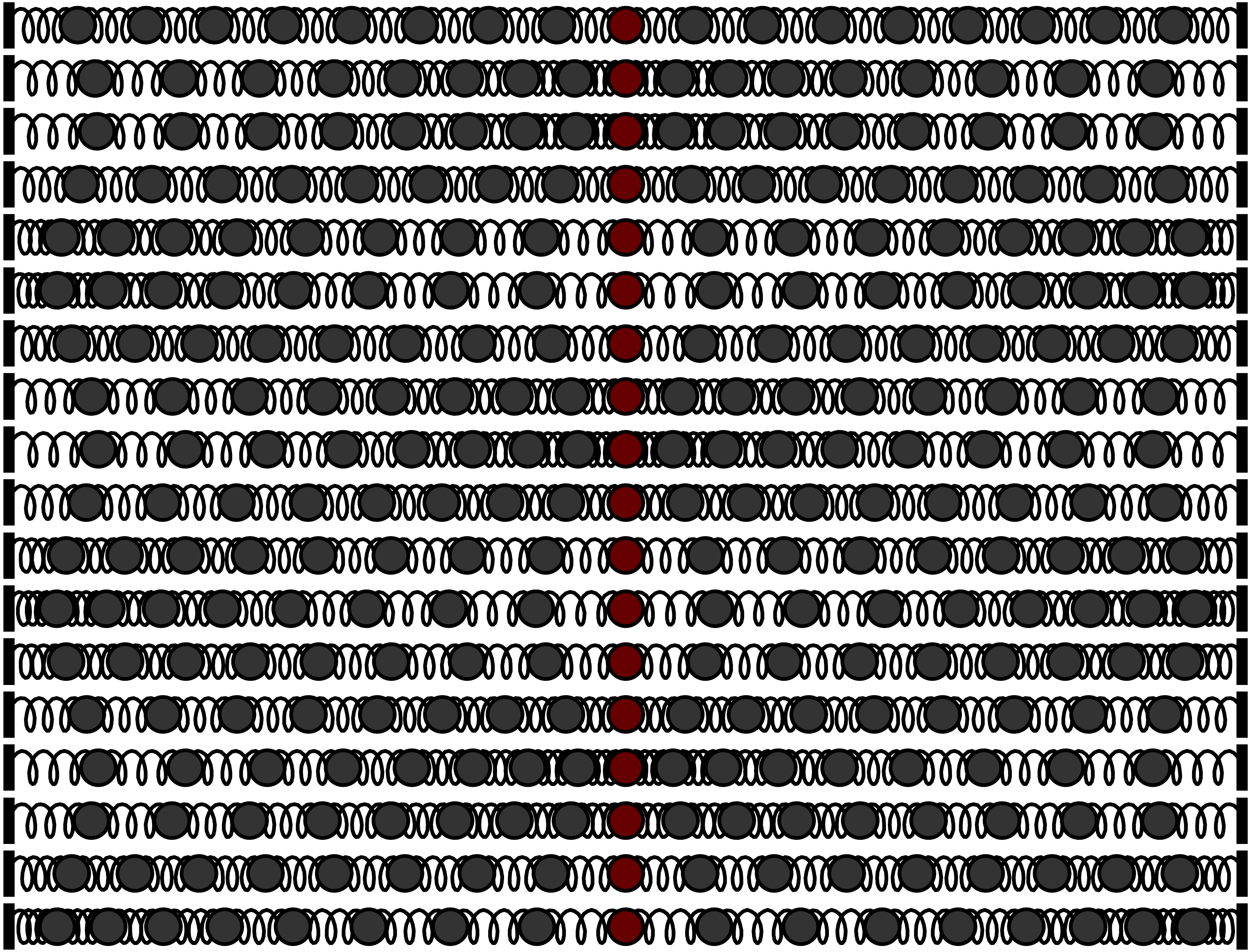
Solutions

$k = 1$



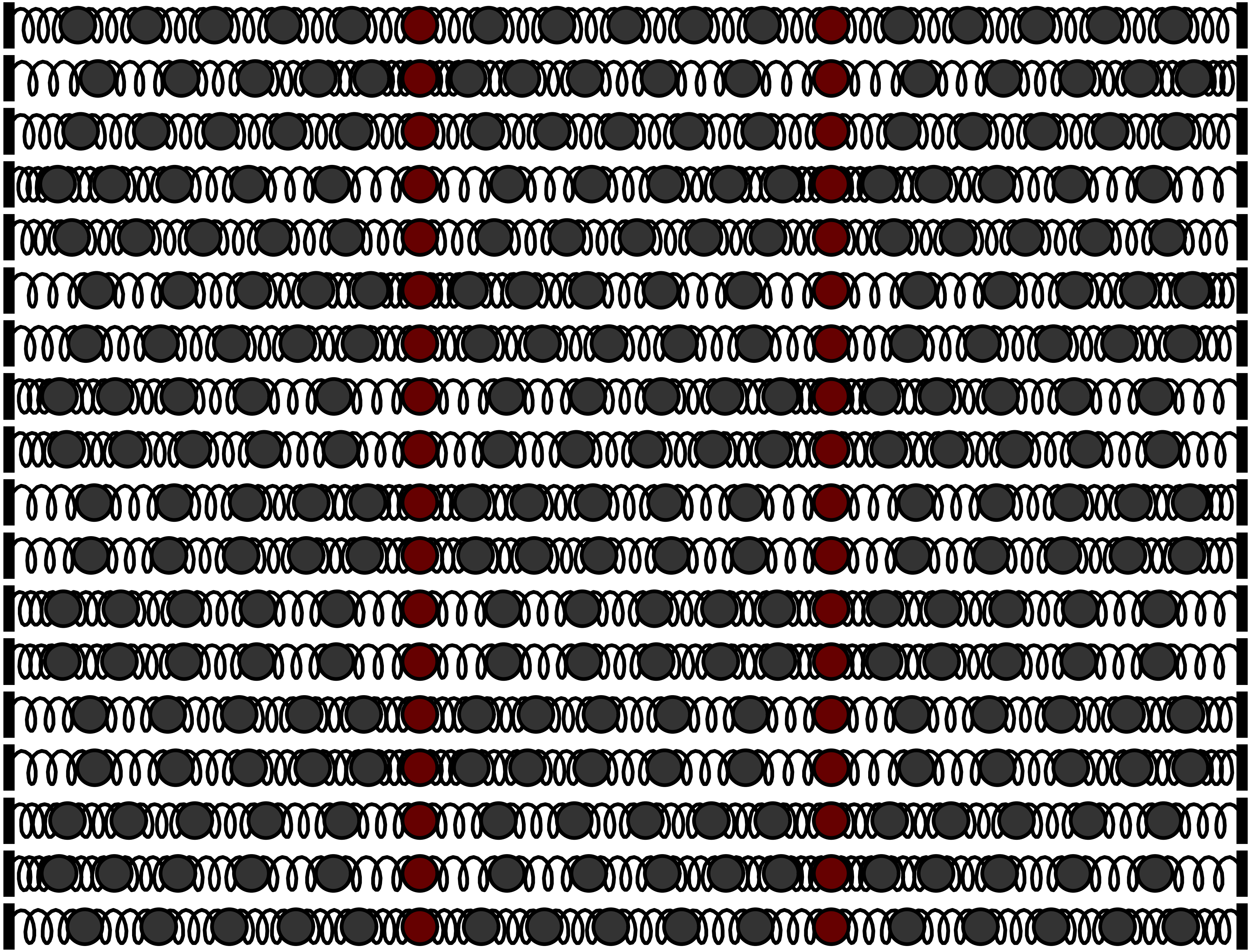
Solutions

$k = 2$

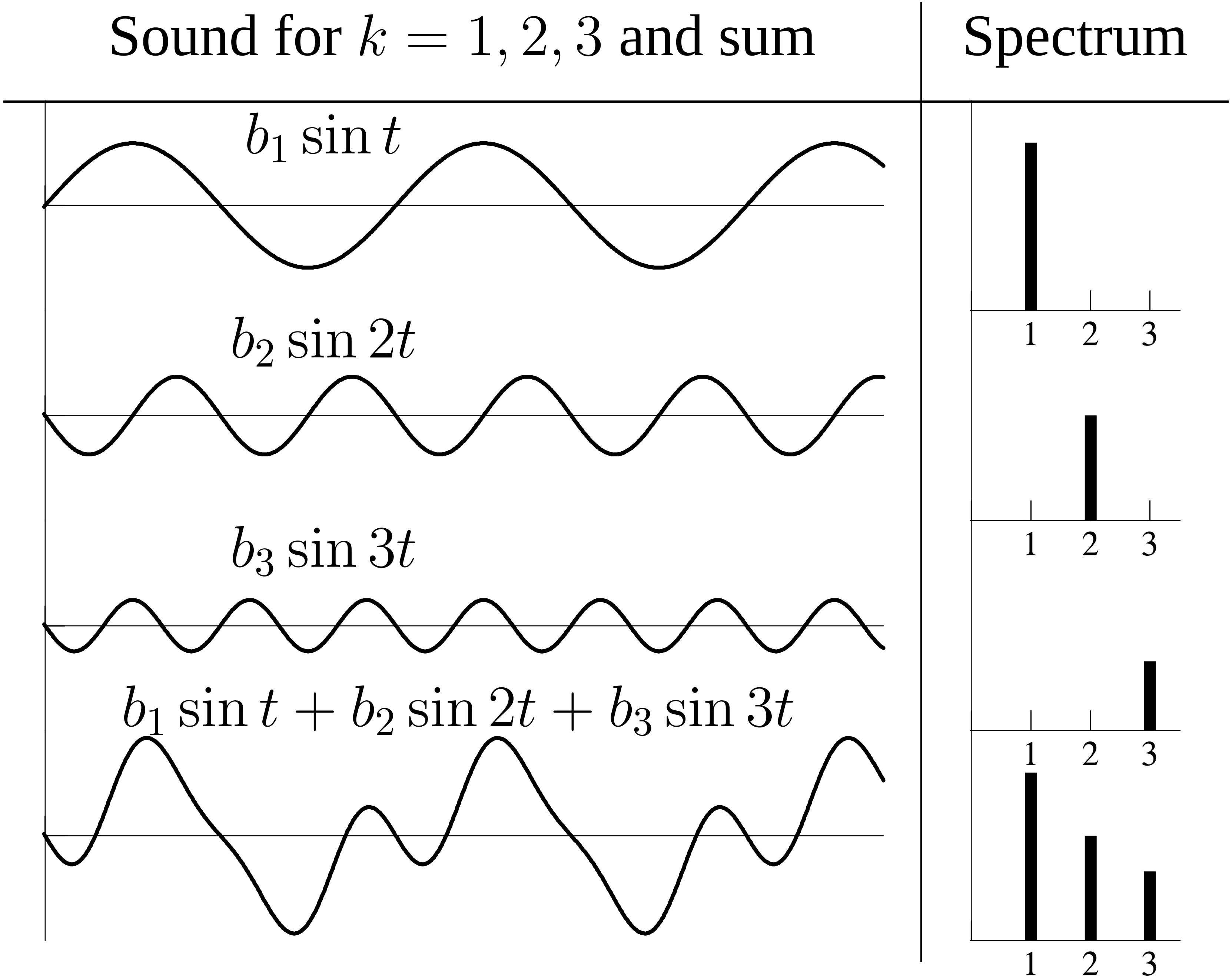


Solutions

$k = 3$



Produced Sound ⇒ Fourier Transform

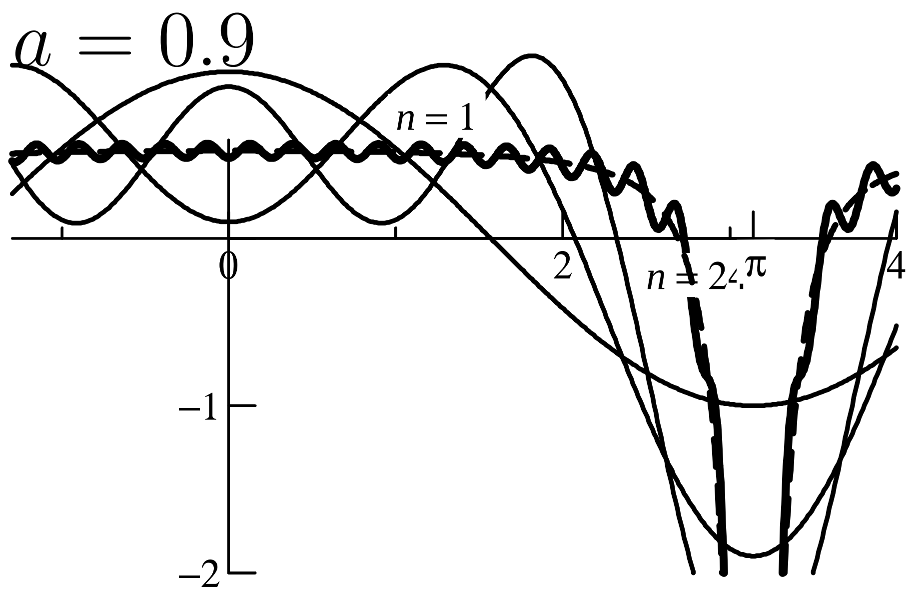


Euler's works on "Fourier Series" E246, E464, E555, E704:

$$z - az^2 + a^2z^3 - a^3z^4 + \ldots = \frac{z}{1+az}$$

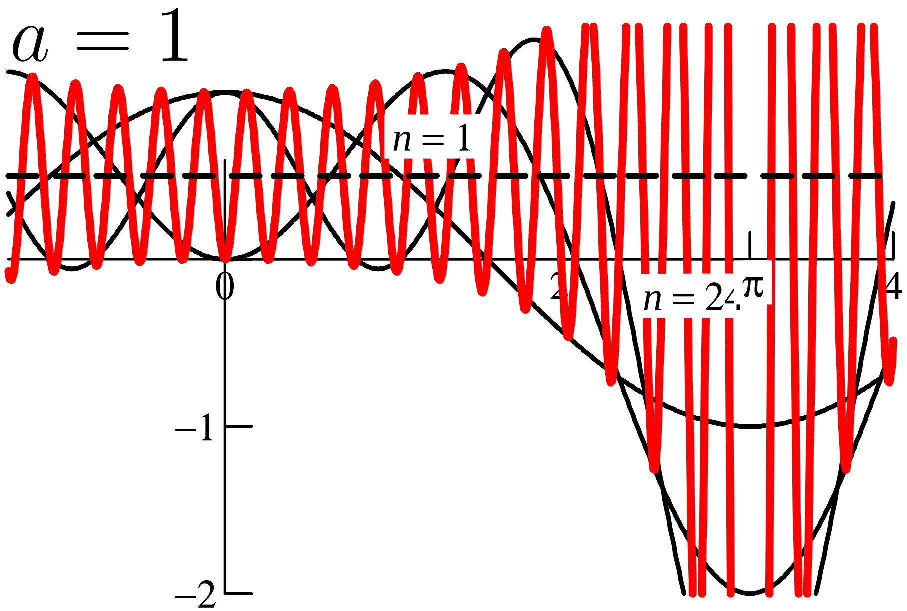
(geometric series, $|a| < 1$). Insert $z = e^{i\varphi}$ and take real part:

$$\cos\varphi - a\cos 2\varphi + a^2\cos 3\varphi - a^3\cos 4\varphi + \ldots = \frac{\cos\varphi + a}{1 + 2a\cos\varphi + a^2}$$



Euler then sets brutally $a = 1$ and writes (nonsense??)

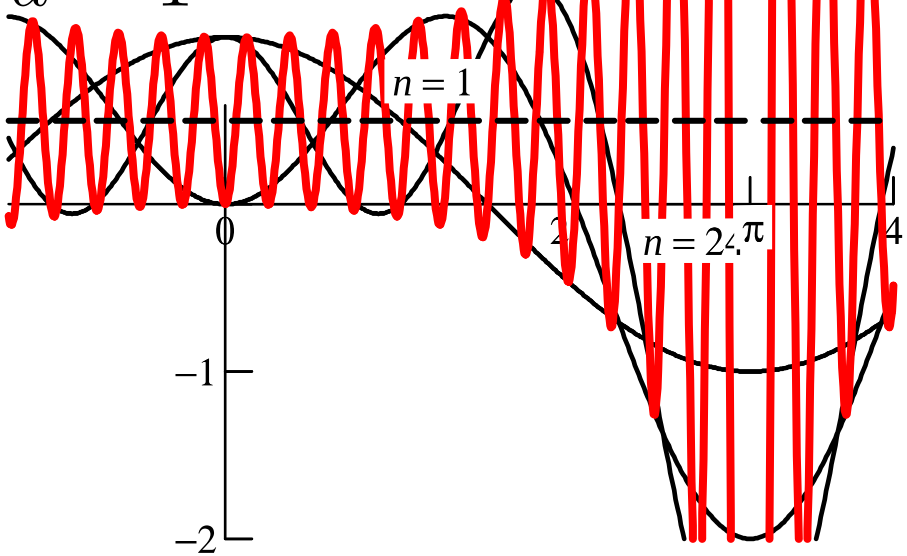
$$\cos\varphi - \cos 2\varphi + \cos 3\varphi - \cos 4\varphi + \ldots = \frac{1}{2}\,.$$



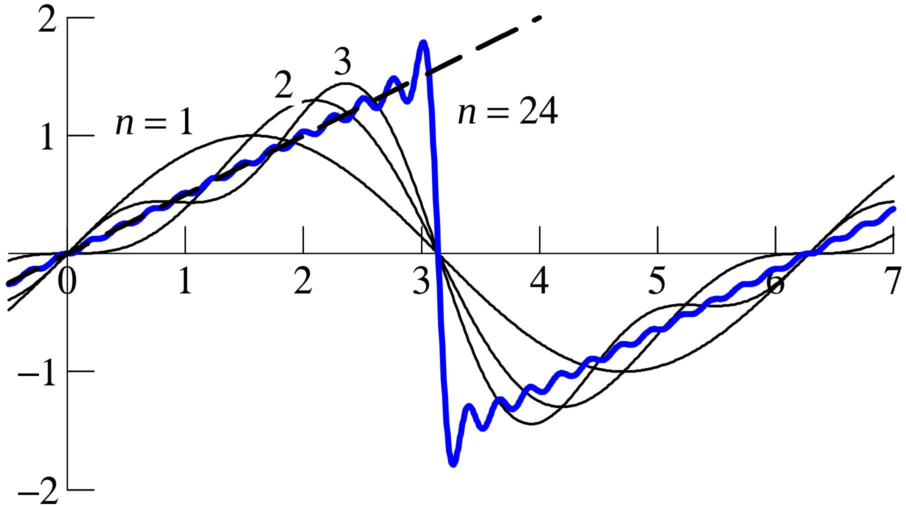
Euler then sets brutally $a = 1$ and writes (nonsense??)

$$\cos\varphi - \cos 2\varphi + \cos 3\varphi - \cos 4\varphi + \ldots = \frac{1}{2}\,.$$

$a = 1$



$a = 1.$, integrated curve



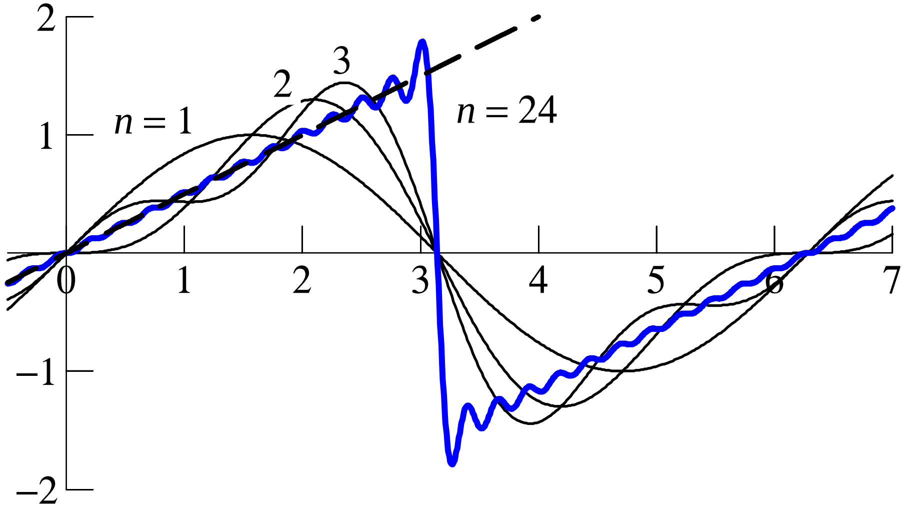
He integrates this formula and obtains

$$\sin\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi - \frac{1}{4}\sin 4\varphi + \ldots = \frac{\varphi}{2}$$

... and the “nonsense” maks again “sense”, two centuries later, in the “sense of distributions”.

$$\sin\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi - \frac{1}{4}\sin 4\varphi + \ldots = \frac{\varphi}{2}$$

$a = 1.$, integrated curve



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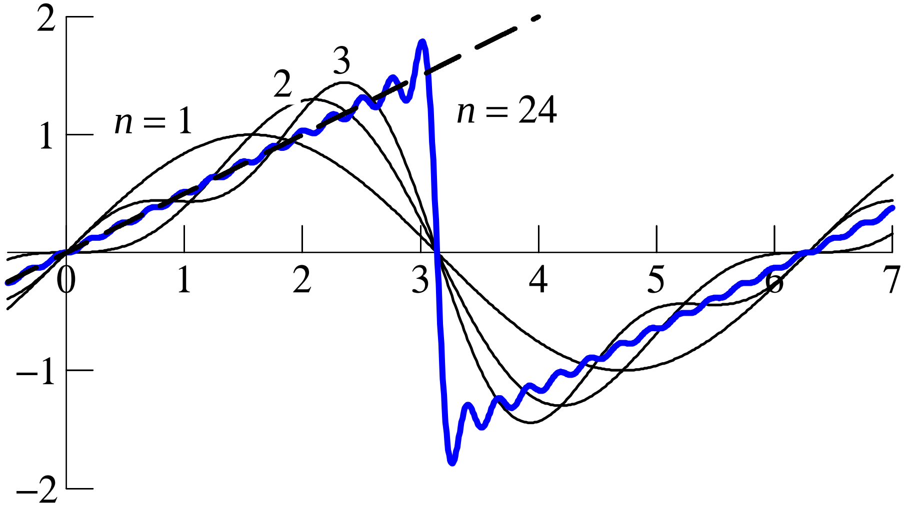
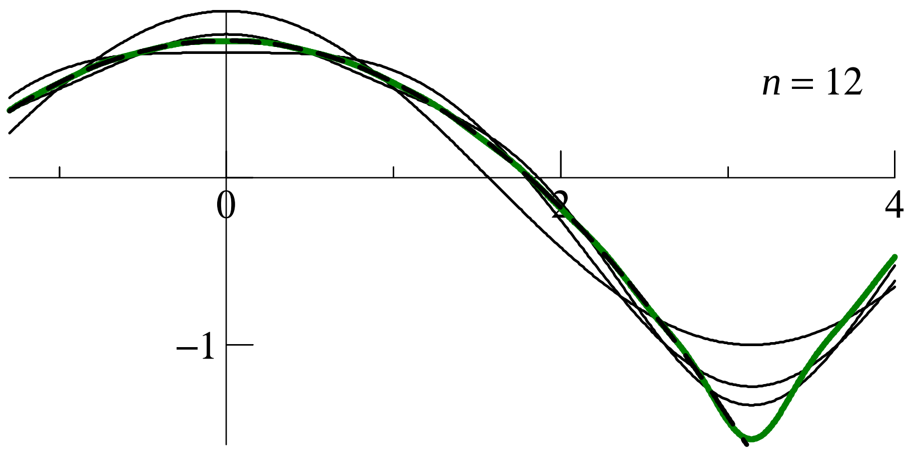


$a = 1.$, 2 times integrated



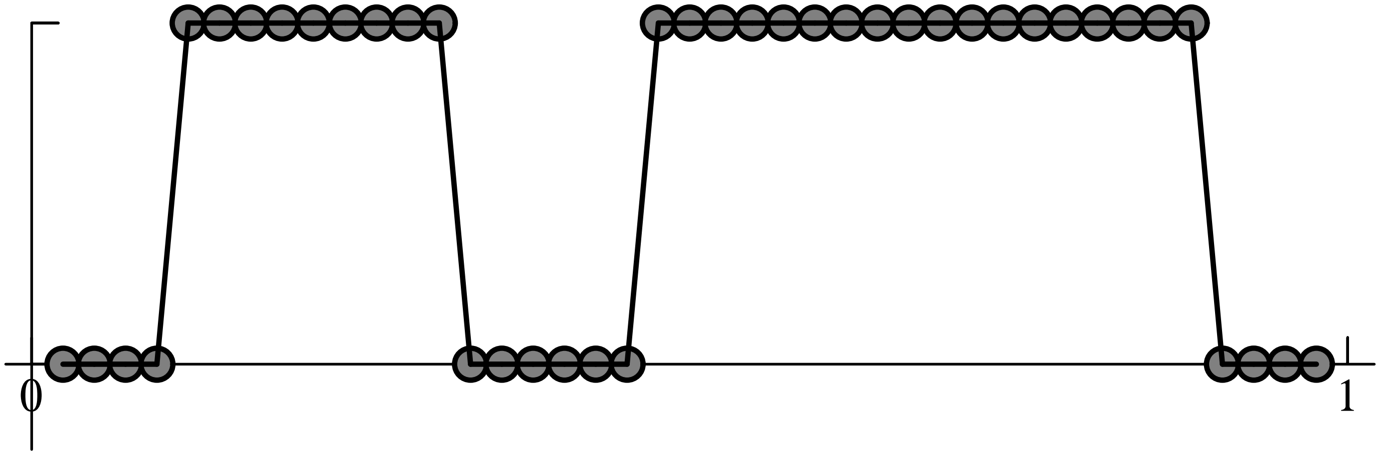
He integrates once more to have

$$\cos\varphi - \frac{1}{4}\cos 2\varphi + \frac{1}{9}\cos 3\varphi - \frac{1}{16}\cos 4\varphi + \ldots = \alpha - \frac{\varphi^2}{4}\,.$$

New acces to Basel problem $(\varphi = 0)$.

Problem. Given data (function $\Phi(\varphi)$), find $b_1, b_2, b_3, \ldots$.

Example.

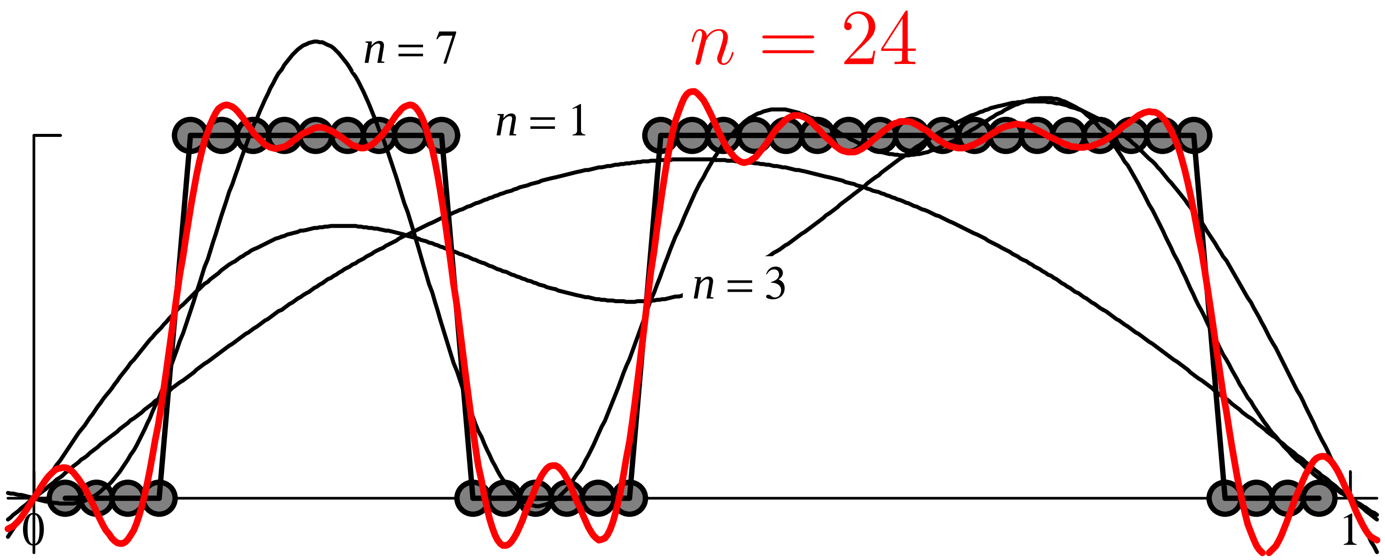


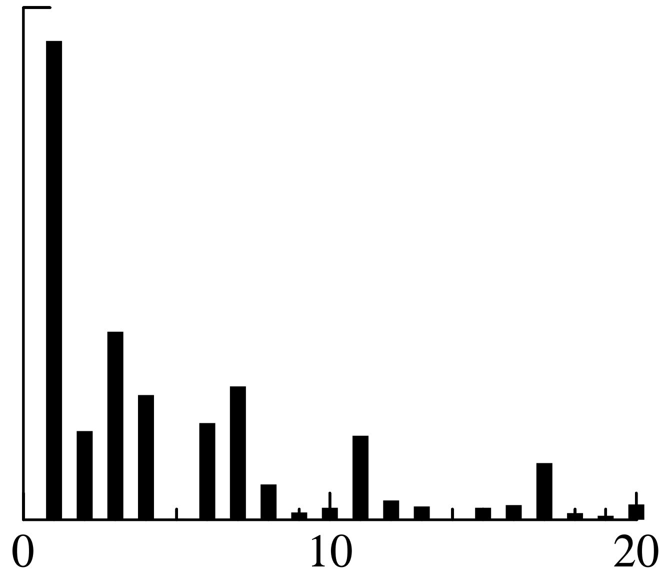
Problem. Given data (function $\Phi(\varphi)$), find $b_1, b_2, b_3, \ldots$.

After decades of calculations (Lagrange on sound, Euler on planetary motion, E703), 70 years old Euler has one of his GREAT ideas (**E704** from 1777, publ. 1798):

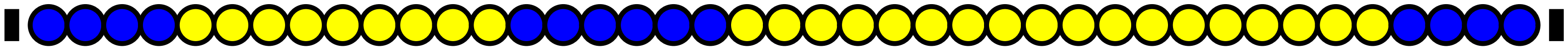
Multiply series $\Phi(\varphi) = b_1 \sin\varphi + b_2 \sin 2\varphi + b_3 \sin 3\varphi + \ldots$ by $\sin k\varphi$, integrate from 0 to π, and use orthogonality to obtain

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin k\varphi \cdot \Phi(\varphi)\, d\varphi \,.$$



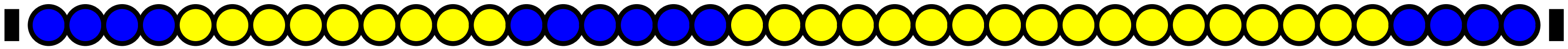


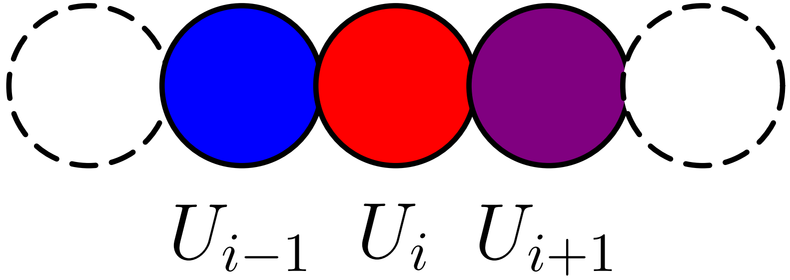
Why is it called Fourier ? **Theory of Heat** (1807, 1822):



Question: How develops heat?

Why is it called Fourier ? **Theory of Heat** (1807, 1822):





Question: How develops heat?

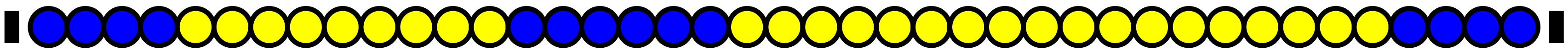
I. Newton:

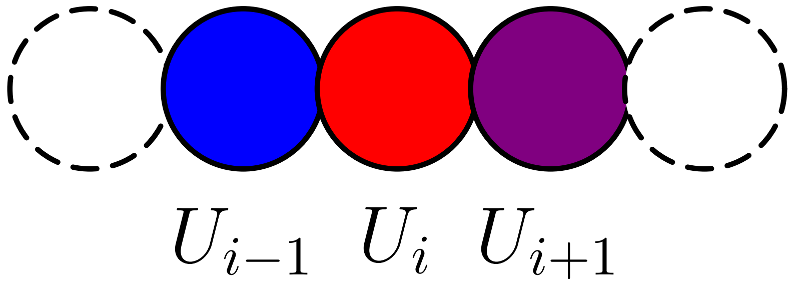
$$\frac{dU_i}{dt} = K \cdot ((U_{i-1} - U_i) + (U_{i+1} - U_i)) = K \cdot (U_{i-1} - 2U_i + U_{i+1})$$

When number of particles $\to \infty$:

$$\frac{\partial u}{\partial t} = a^2 \, \frac{\partial^2 u}{\partial x^2}$$ (Fourier's heat equation).

Why is it called Fourier ? Theory of Heat (1807, 1822):





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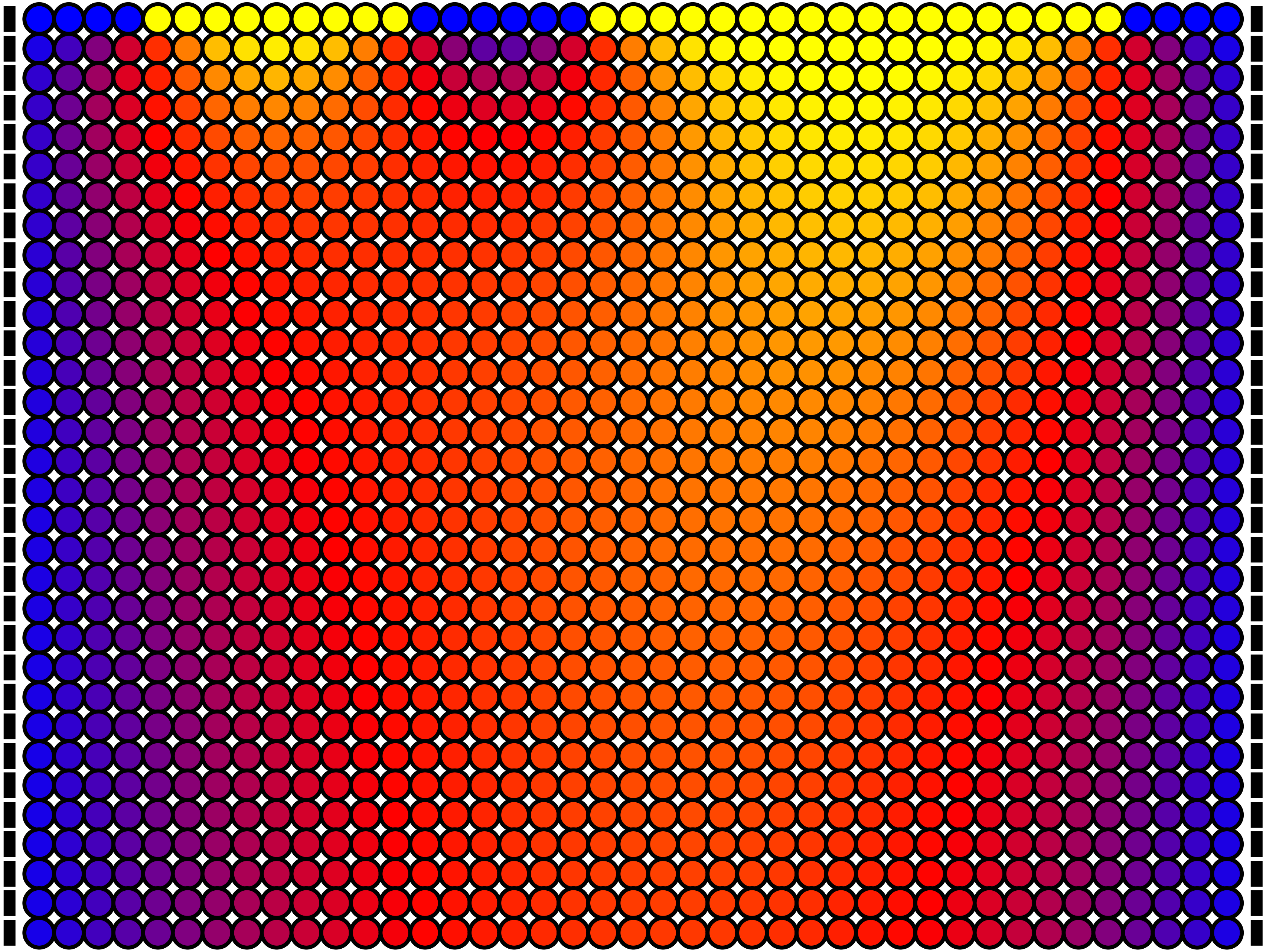
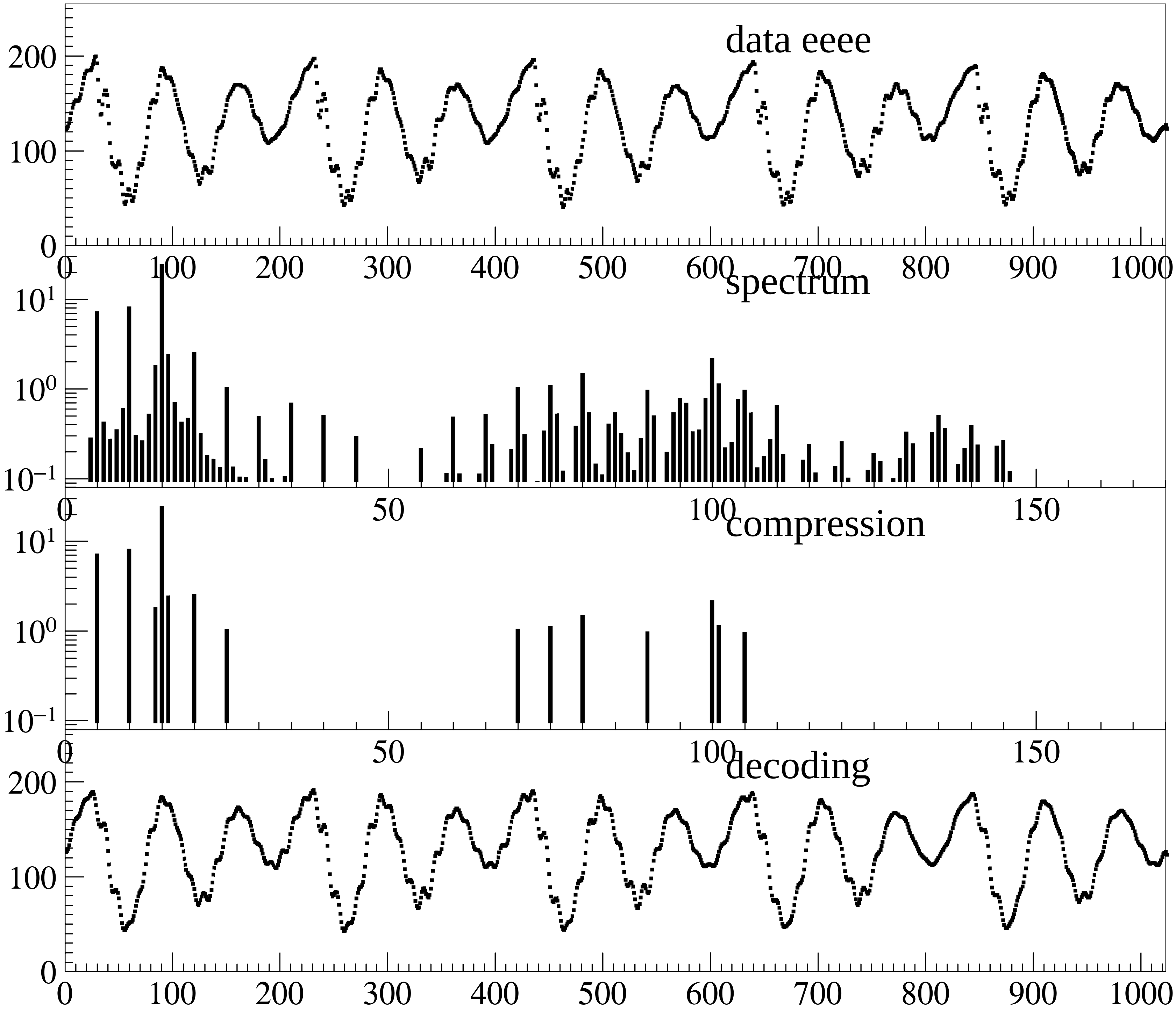
Solution:

$$\underbrace{\frac{\partial u}{\partial t}}_{-a^2k^2u} = a^2 \, \underbrace{\frac{\partial^2 u}{\partial x^2}}_{-k^2 u}$$ (as above) $\Rightarrow$ $u = \sum_k b_k \sin(kx) \cdot e^{-a^2k^2t}$.

Computed Solution :



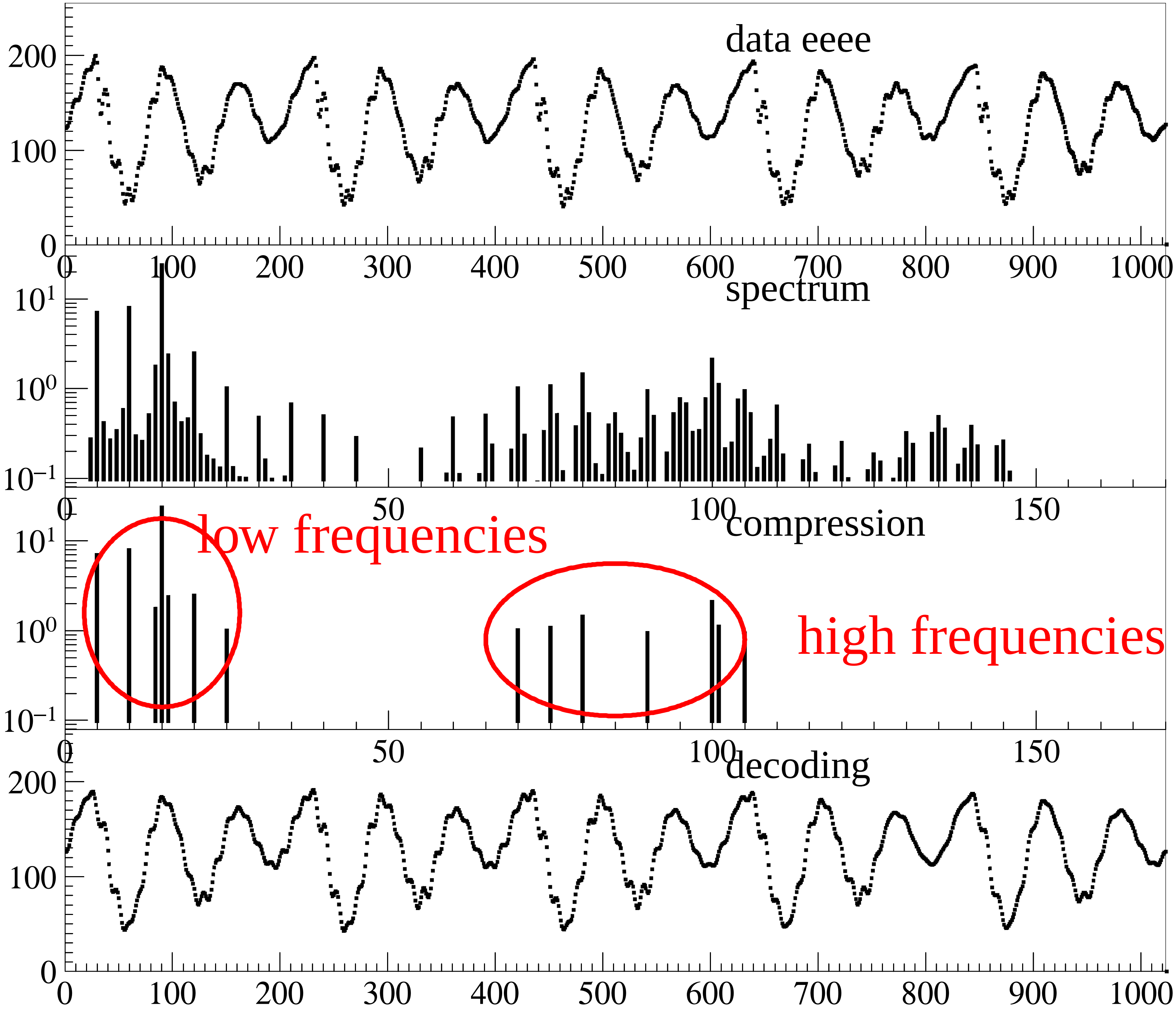
Principle of data compression (MP3). (Kh. Brandenburg)



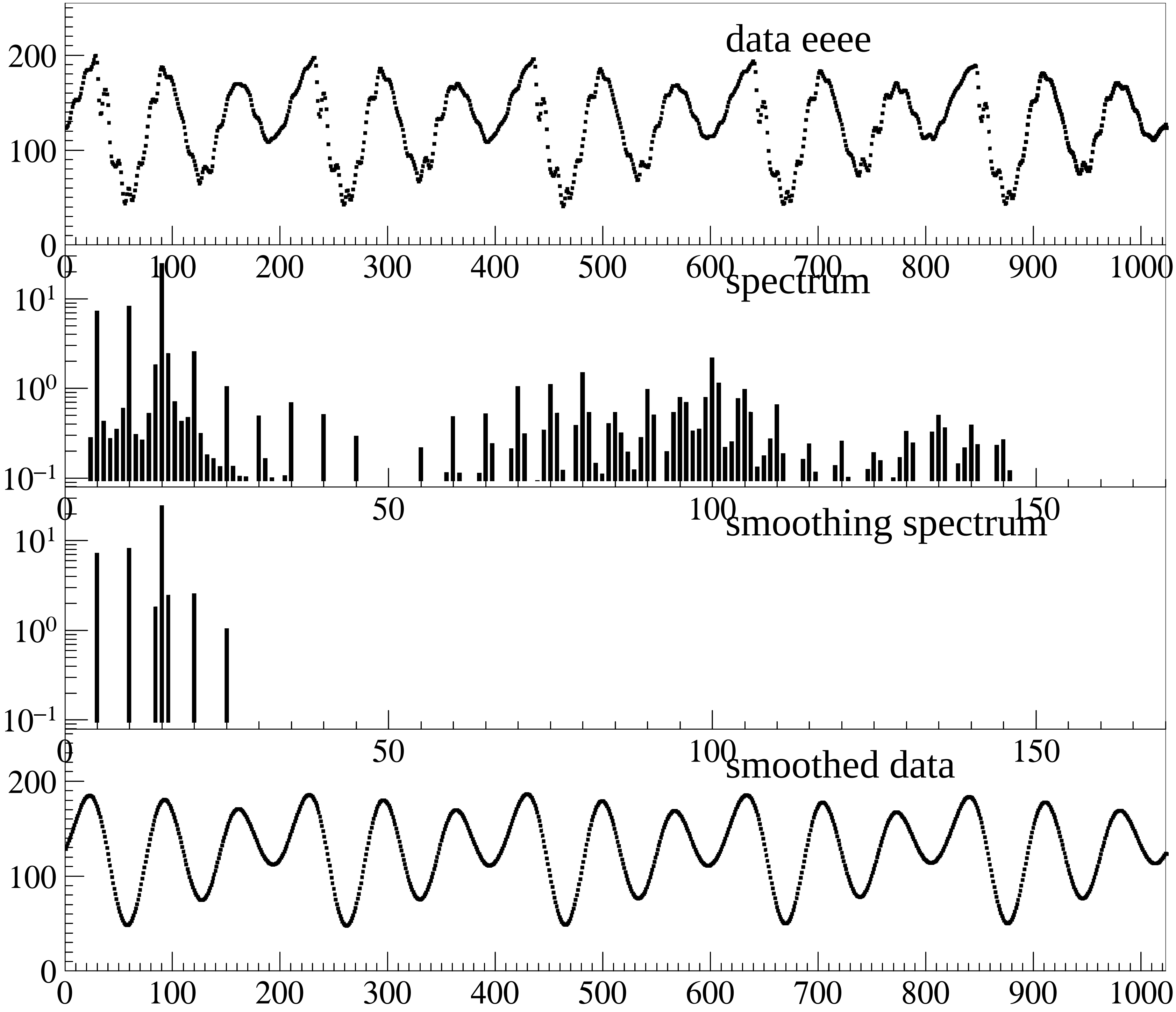
- Converting sound samples to frequency data is the single most common task of all DSP chips.
- CDs encode sound as linear PCM (pulse code modulation), hence, not compressed [1981 technology] !
- Transferring CD tracks to something like an Apple iPod (portable music device) will compress the results from the FFT data.
- Audio tracks on DVDs use a similar process, compressing sound frequency data.

(Thanks to Kyle Granger for these informations).

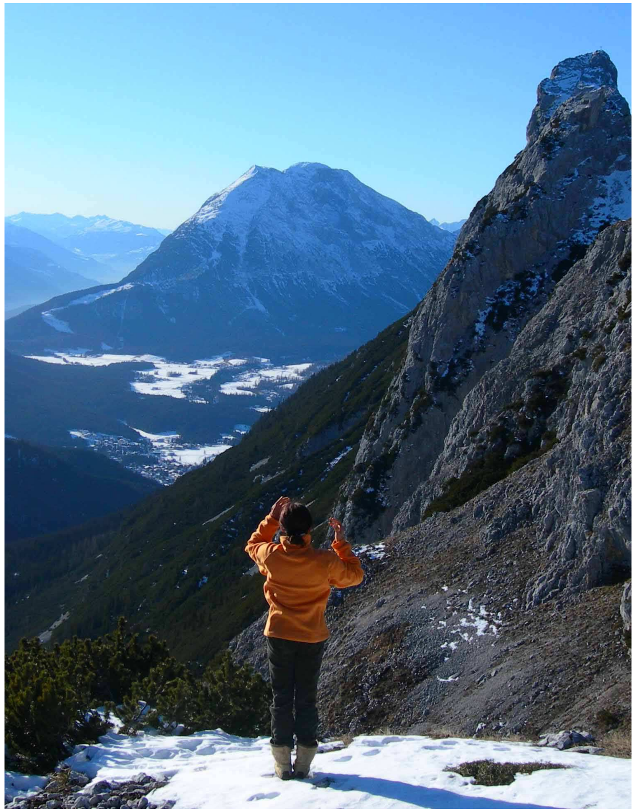
Principle of data smoothing, denoising.



Principle of data smoothing, denoising.



Principle of JPEG data compression.



any photograph

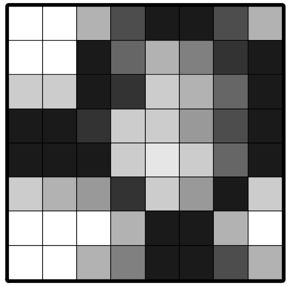
is cut into pieces

of 8×8 pixel

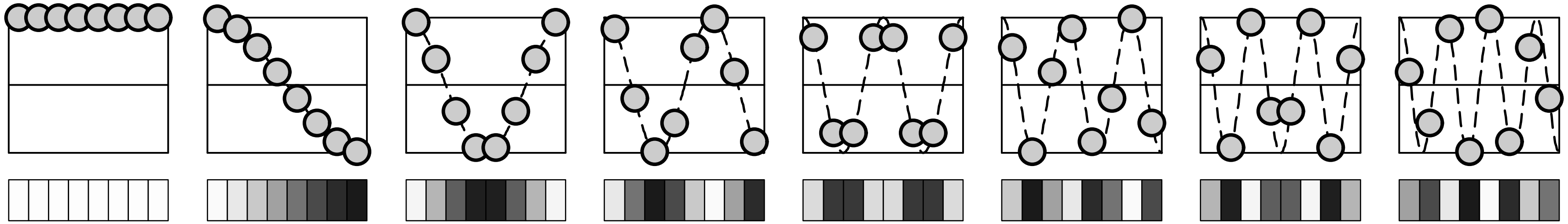
$\Rightarrow$

and each piece is

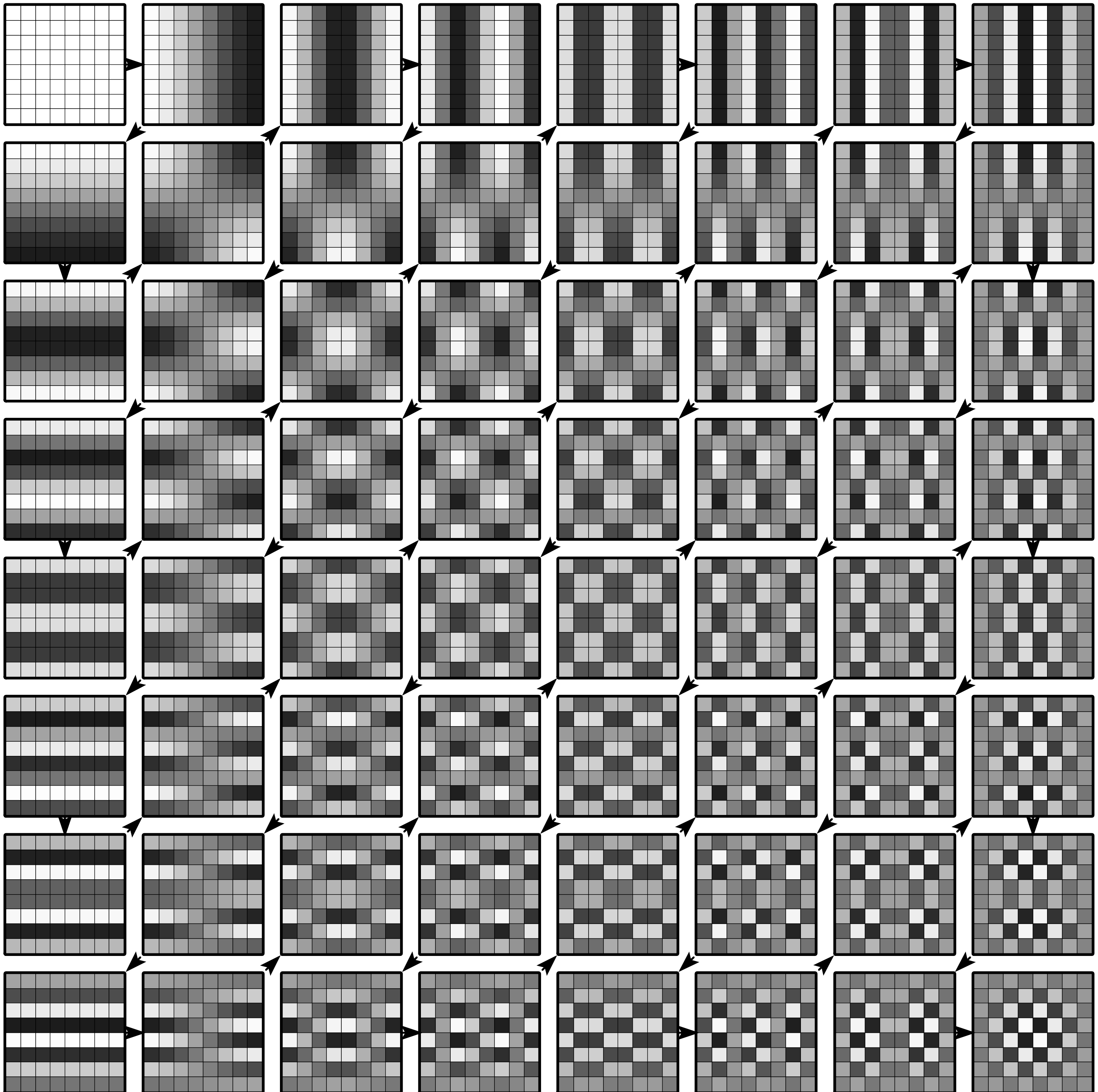
treated individually



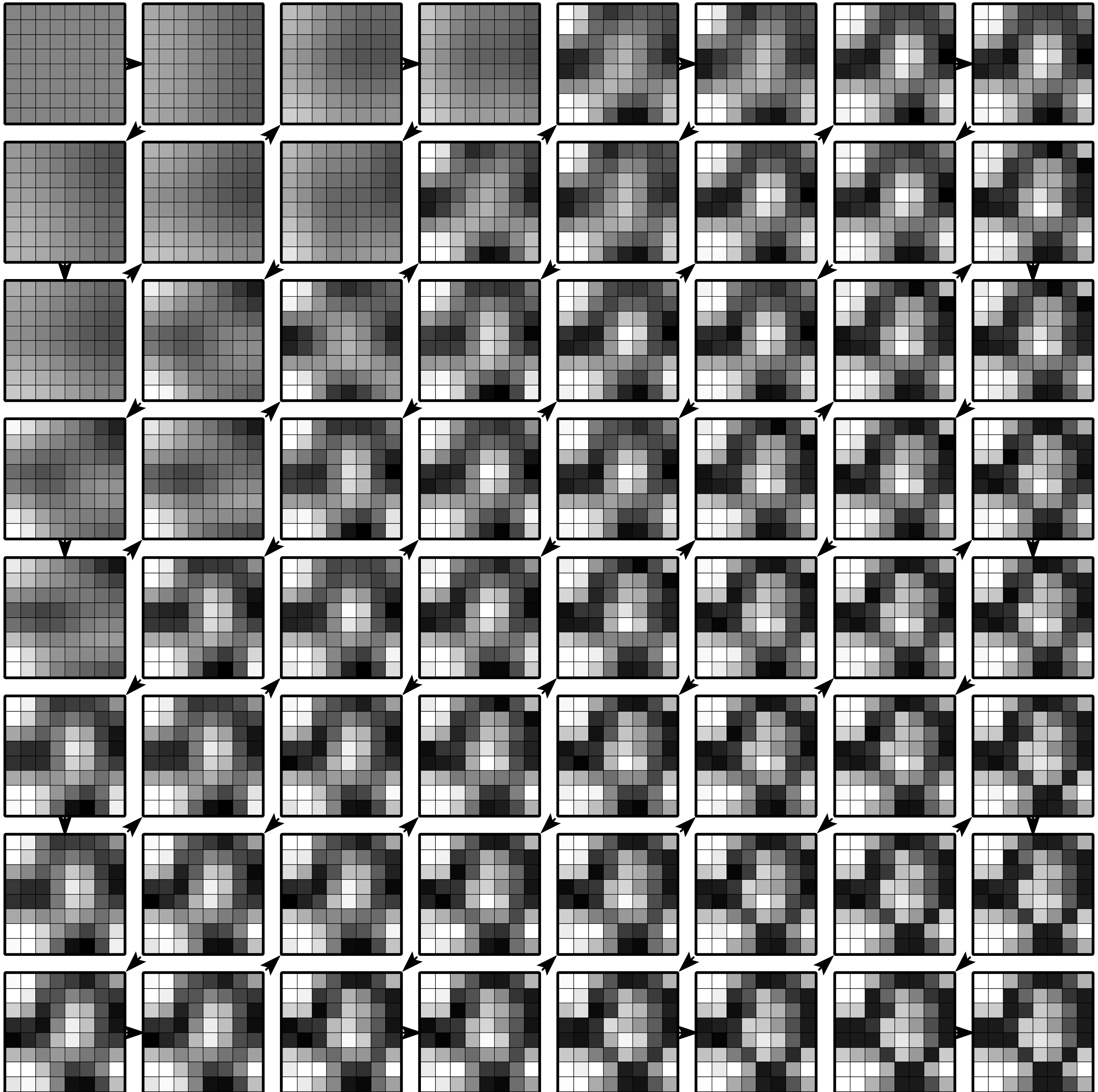
"Sound" is now two-dim.! $\Rightarrow$ one-dim. base $\cos k\pi x$



transform into two-dimensional base... $\cos k\pi x \cdot \cos \ell\pi y$:



stages of reconstruction of image :



Animations :

1. Sound transmission (animation by Gilles Vilmart);
2. Sound analysis by Amadeus II (program by Martin Hairer).