Applied Maths and Mechanics

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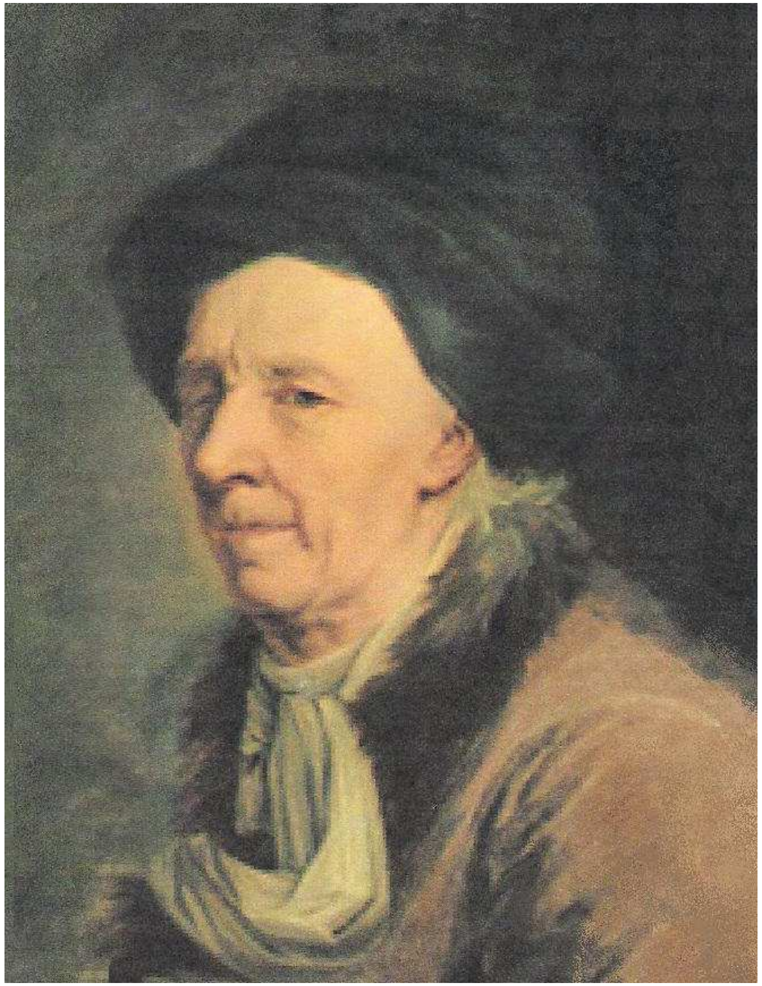
1707  1783

Ph. Henry & G. Wanner

Bellinzona Agosto 2007.

Applied Mathematics.

> “Weilen nun die H. Staatsministri fleissig zugegen sind, so habe, um diesen Herren keinen Ekel zu erwecken, meine Dissertationen französisch abgelesen, (...). Ich habe auch um dieser Ursach willen pure mathematische Spekulationen und calculos zu evitieren gesucht und mehrenteils physikalische Materien abgehandelt.”
>
> (One of the first letters of Euler to Goldbach from — ?? — from Berlin (4. Jul. 1744))

Euler’s greatest works on applied mathematics appeared mainly during his stay in Berlin (1741-1766, see citation).

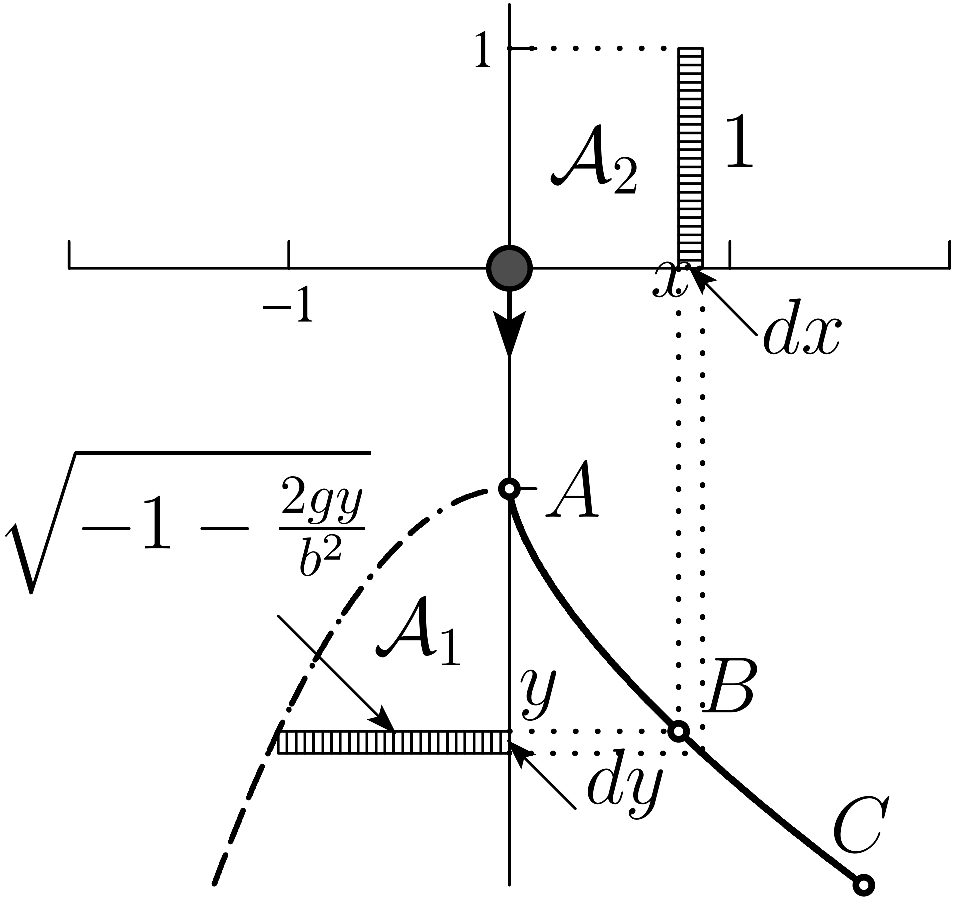
The Bernoulli Brothers.

"(...) en 1684, la face de la Géométrie changea presque tout à coup. L'illustre M. Leibenits donna (...) quelques essais de son nouveau calcul Différentiel (...), dont il cachoit l'art & la méthode. Aussi-tôt Mrs Bernoulli (...) sentirent par le peu qu'ils voyoient de ce calcul qu'elle en devoit être l'étenduë & la beauté, ils s'appliquerent opiniâtrement à en chercher le secret, & à l'anlever à l'inventeur, ils y réussirent, & perfectionnerent cette mèthode au point que M. Leibenits par une sincerité digne d'un grand homme, a déclaré qu'elle leur appartenoit autant qu'à lui. C'est ainsi que le moindre rayon de verité qui s'échape au travers de la nuë, éclaire suffisament les grands esprits, tandis que la verité entièrement dévoilée, ne frappe pas les autres."

(Fontenelle, *Eloge de Jakob Bernoulli*, 1740)

Jakob's Solution of the Isochrone. (1690)

($\iota\sigma o\varsigma$ =equal, $\chi\rho o\nu o\varsigma$ =time)



know: $$(\frac{ds}{dt})^2 = -2gy$$

want: $$(\frac{dy}{dt})^2 = b^2$$

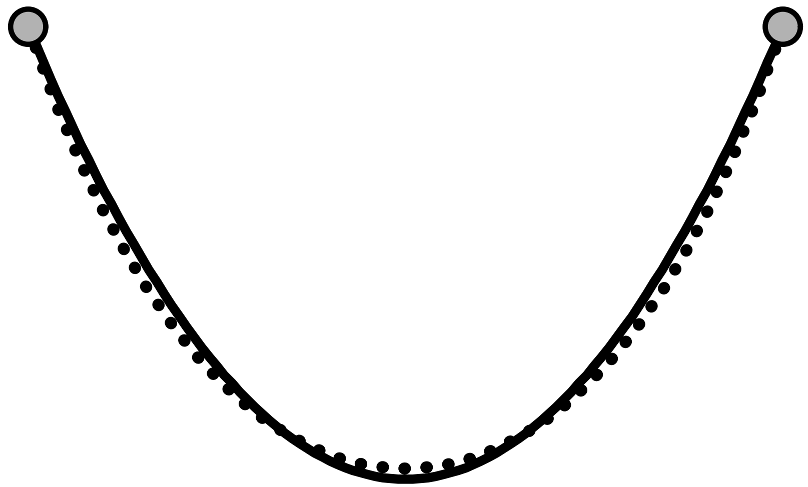
divide: $$dx = -\sqrt{-1-\frac{2gy}{b^2}}\, dy.$$

"ergo & horum integralia æquantur"

$$x = \frac{b^2}{3g}(-1-\frac{2gy}{b^2})^{3/2}\,.$$

first use of term "integral", first practical differential equation solved, "by separation of variables".

The Catenary. (Galilei wrong, Jakob pose, Johann solve 1691)

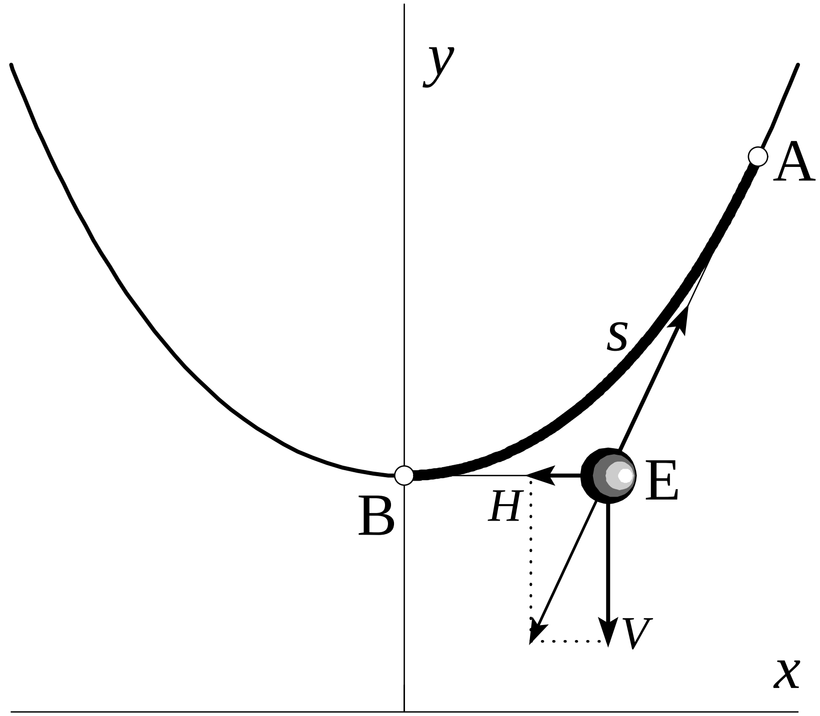


"Les efforts de mon frere furent sans succès, pour moi, je fus plus heureux, car je trouvai l'adresse (...). Il est vrai que cela me couta des meditations qui me deroberent le repos d'une nuit entiere (...)."

"Je ne mets point ici la démonstration, parce que ceux qui entendent ces matiéres, la trouveront aisément, & qu'il faudroit trop de discours pour la faire comprendre aux autres."

(Joh. Bernoulli)

Johann's Solution.



$$c\,p = s \quad \text{with} \quad p = \frac{dy}{dx}$$

$$c\,dp = ds = \sqrt{1+p^2}\,dx\,.$$

$$c\int \frac{dp}{\sqrt{1+p^2}} = \int dx$$

$$\operatorname{arsinh}(p) = \frac{x-x_0}{c}$$

$$p = \sinh(\frac{x-x_0}{c})$$

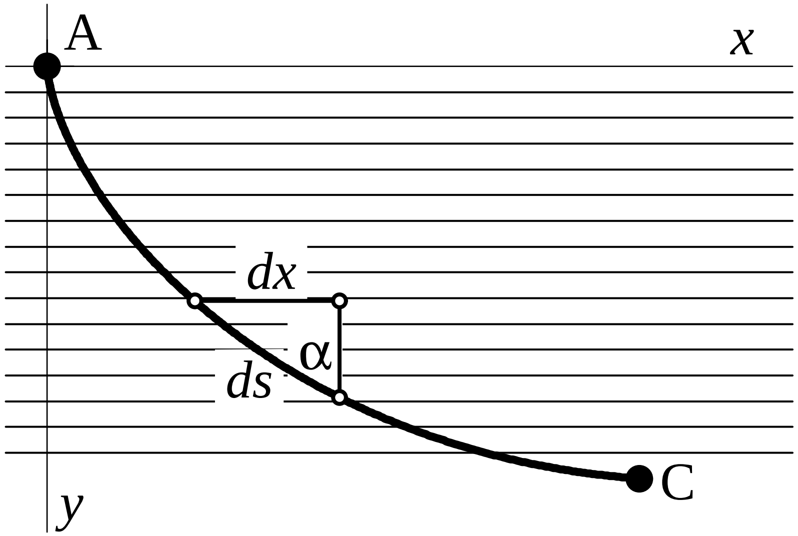
$$y = K + c\cosh(\frac{x-x_0}{c})\ .$$

The Brachystochrone . ($\beta\rho\alpha\chi\upsilon\varsigma$ =short, $\chi\rho o\nu o\varsigma$ =time).

Solution of Johann (1697):

$$\frac{v}{\sin\alpha} = K \quad (\sin\alpha = \frac{1}{\sqrt{1+p^2}})$$

$$\sqrt{1+p^2}\cdot\sqrt{2gy} = K$$



$$\text{or} \quad dx = \sqrt{\frac{y}{c-y}}\cdot dy$$

$$\text{where } c = K^2/2g$$

$$\text{subst.} \quad y = c\cdot\sin^2 u = \frac{c}{2} - \frac{c}{2}\cos 2u$$

$$\text{gives} \qquad x - x_0 = c\,u - \frac{c}{2}\sin 2u\ .$$

‘ex qua concludo curvam Brachystochronam esse cycloïdem vulgarem”.

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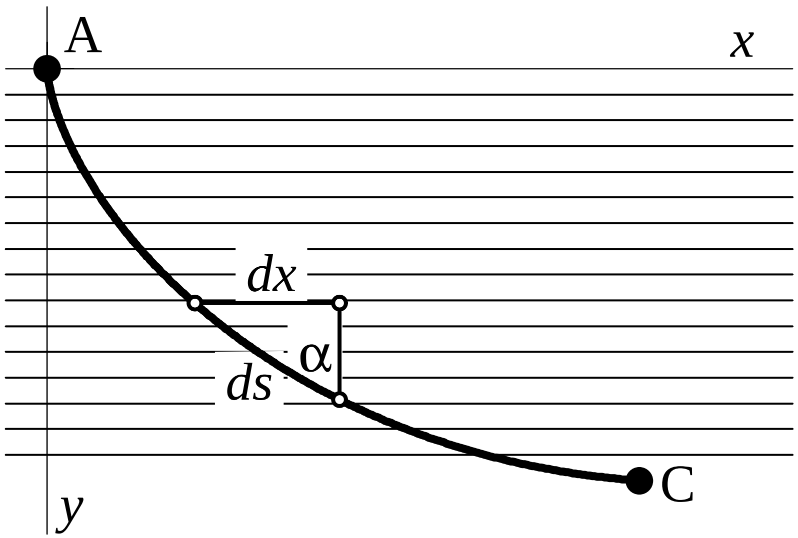


or $dx = \sqrt{\dfrac{y}{c-y}}\cdot dy$

where $c = K^2/2g$

subst. $y = c\cdot\sin^2 u = \dfrac{c}{2} - \dfrac{c}{2}\cos 2u$

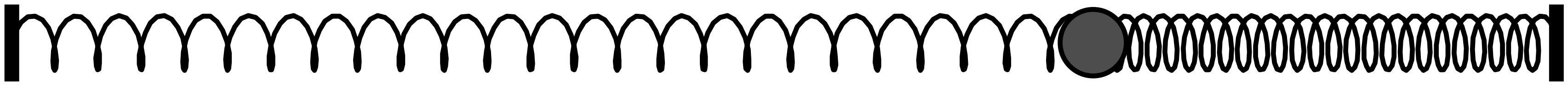
gives $x - x_0 = c\,u - \dfrac{c}{2}\sin 2u$.

‘ex qua concludo curvam Brachystochronam esse cycloïdem vulgarem”.

All these were differential equations of first order.

Dynamics.

Problem. How moves a mass point attached to springs ?



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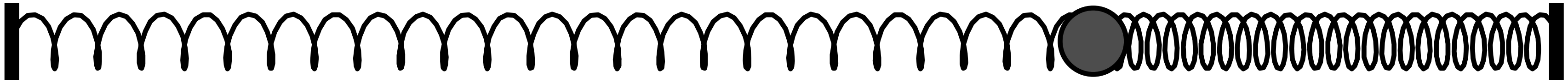
Problem. How moves a mass point attached to springs ?



R. Hooke (1678): Force $f = -K \cdot y$.

I. Newton (1687): "Change of movement is proportional to the acting force" (orig. in Latin); in formulas:

$$m \cdot \dot{v} = f \qquad \dot{y} = v\ .$$

L. Euler (E15, 1736): Have to solve differential equation

$$m \cdot \ddot{y} + K \cdot y = 0 \qquad \text{or} \qquad \frac{ddy}{dt^2} + k^2 y = 0$$

of second order!!

Higher Order Diff. Equations (E10 (1728), E62 (1739)).

$$0 = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2} + \frac{D\,d^3y}{dx^3} + \ldots + \frac{N\,d^ny}{dx^n}$$

si ponamus $y = c^v$, later

$$y = e^{\int p\,dx} \quad \Rightarrow \quad \begin{aligned} \frac{dy}{dx} &= e^{\int p\,dx} \cdot p \\ \frac{ddy}{dx^2} &= e^{\int p\,dx}(pp + \frac{dp}{dx}) \\ \frac{d^3y}{dx^3} &= e^{\int p\,dx}(p^3 + \frac{3\,p\,dp}{dx} + \frac{ddp}{dx^2}) \end{aligned}$$

... atque remanebit aequatio differentialis gradus $n - 1$.

- Opens treatment of mechanical problems by diff. equs.;
- If $A, B, C \ldots$ constants, $\Rightarrow$ $p(x)$ const. $\Rightarrow$ $y = e^{px}$.

For our problem: E62, §22:

$$y'' + y = 0 \Rightarrow \quad p^2 + 1 = 0 \quad \Rightarrow \quad p = \pm\sqrt{-1} = \pm\, i$$

quibus coniunctis fit $\quad y(x) = \gamma e^{+ix} + \delta e^{-ix}$.

Other method E62, §18: Euler's multiplyer method:

$$\text{(multiply:)} \qquad 2y'y'' + 2y'y = \frac{d}{dx}(y'^2 + y^2) = 0$$

$$\Rightarrow y'^2 + y^2 = \alpha^2, \quad y' = \sqrt{\alpha^2 - y^2}, \quad \int \frac{dy}{\sqrt{\alpha^2 - y^2}} = \int dx,$$

$$\arcsin\frac{y}{\alpha} = x + \beta\,. \qquad \Rightarrow \qquad y(x) = \alpha \cdot \sin(x + \beta)\,.$$

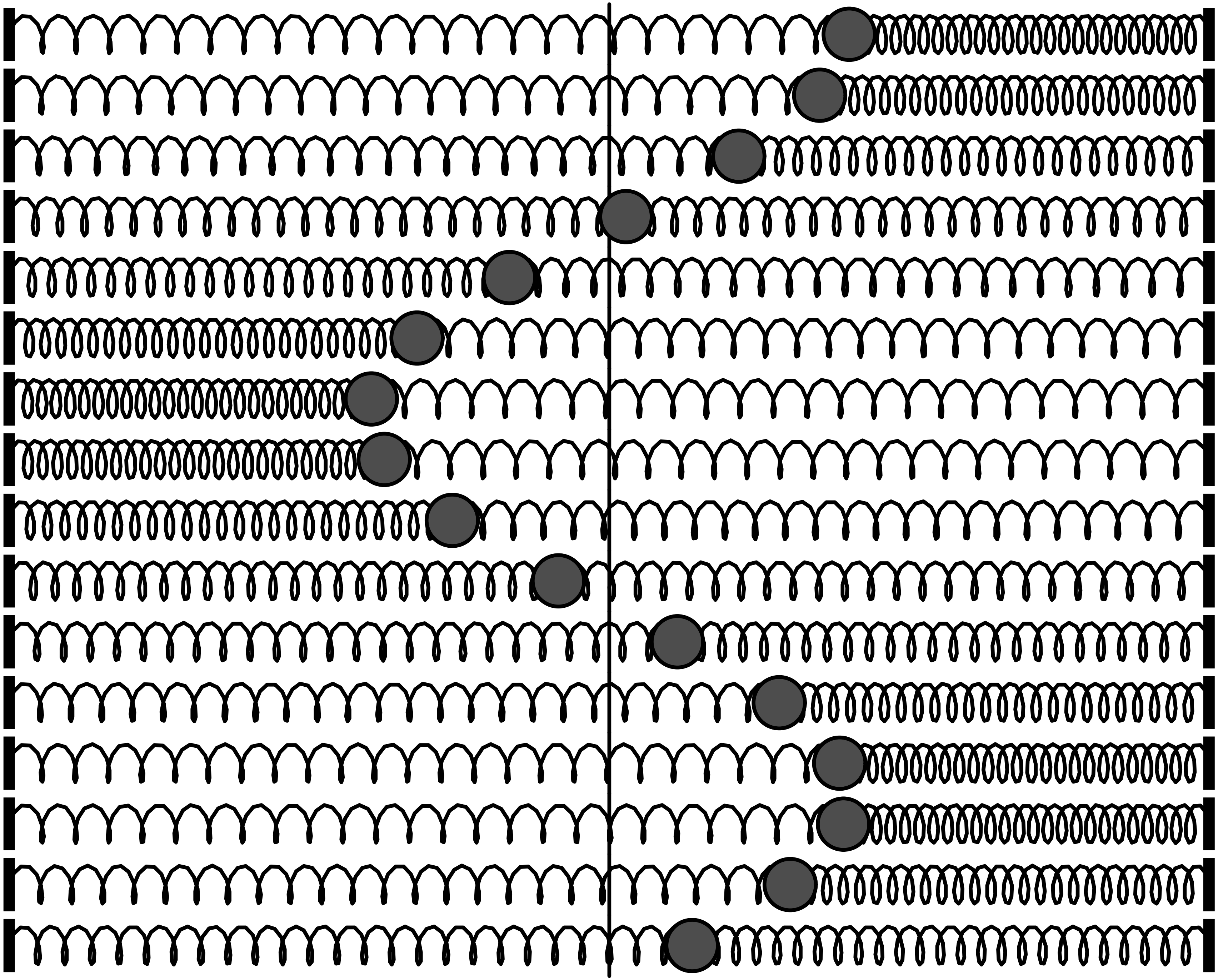
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x + i\sin x = e^{ix}$$

“autem exponentialibus in series conversis”

use $i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ \ldots$:

$$
\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \ldots \\
&= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \ldots\right)}_{\cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \ldots\right)}_{\sin x} \\
&= \cos x + i \sin x.
\end{aligned}
$$

Solution.



Mechanica sive motus scientia analytice exposita (E15).

General principle: Euler replaces Newton's geometrical argumentations by "analytical" differential equations.

Example: The proof of Kepler's second law:

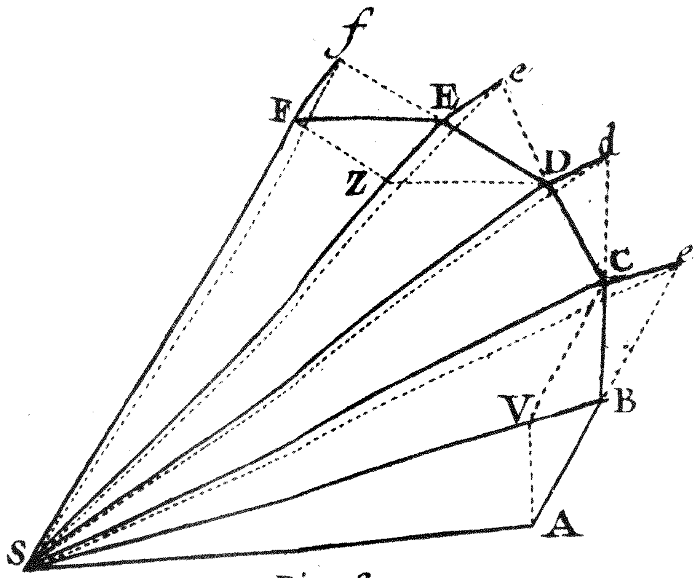


Fig. 13.

Newton, *Principia*

$$ddx = -2g\,dt^2\,\frac{Ax}{v^3}$$

$$ddy = -2g\,dt^2\,\frac{Ay}{v^3}$$

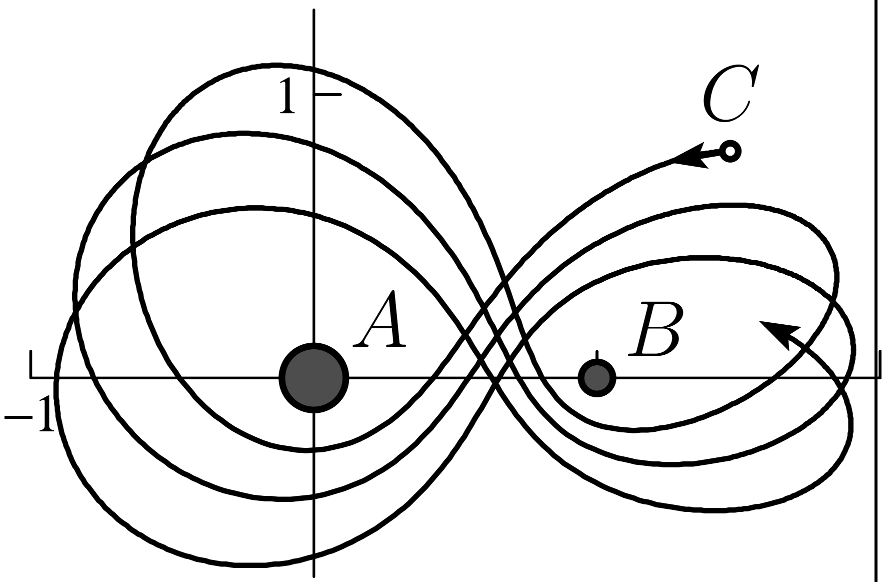
$$v = \sqrt{xx + yy}$$

$$x\,ddy - y\,ddx = d(x\,dy - y\,dx) = 0\,.$$

Euler (notation of E301)

Advantage. More complicated problem ...
$\Rightarrow$ just add more terms to diff. equation ...
... and try to solve ...

Example: Two fixed centers: (E301, E337) from 1760.



$$\frac{ddx}{dt^2} = -\frac{Ax}{v^3} - \frac{B(x-a)}{u^3}$$

$$\frac{ddy}{dt^2} = -\frac{Ay}{v^3} - \frac{By}{u^3}$$

$$v = \sqrt{x^2+y^2}, \quad u = \sqrt{(x-a)^2+y^2}$$

$$A = 2, \quad B = 1, \quad a = 1$$

$$x_0 = 1.47, \ \dot{x}_0 = -0.81,$$

$$y_0 = 0.8, \ \dot{y}_0 = 0.$$

Numerical Method (E342).

(Inst. Calc. Integr. 1768, Vol. I, Sect. Secunda §650).

Ipsius	valores successivi										
x	a,	a',	a'',	a''',	a^{IV},	.	.	.	.	$'x$,	x
y	b,	b',	b'',	b''',	b^{IV},	.	.	.	.	$'y$,	y
V	A,	A′,	A″,	A‴,	A^{IV},	.	.	.	.	′V,	V.

$$b' = b + A(a' - a),\ \ b'' = b' + A'(a'' - a'),\ \ b''' = b'' + A''(a''' - a''),\ \ldots$$

§656: Taylor method. Example: $y' = x^2 + y^2$:

Exemplum 2.

662. *Aequationis differentialis* $\partial y = \partial x (xx + yy)$ *integrale completum proxime investigare.*

Cum hic sit $\frac{\partial y}{\partial x} = V = xx + yy$, erit continuo differentiando

$$\frac{\partial\partial y}{\partial x^2} = 2x + 2xxy + 2y^3 \text{ et}$$

$$\frac{\partial^3 y}{\partial x^3} = 2 + 4xy + 2x^4 + 8xxyy + 6y^4$$

$$\frac{\partial^4 y}{\partial x^4} = 4y + 12x^3 + 20xyy + 16x^4y + 40xxy^3 + 24y^5$$

$$\frac{\partial^5 y}{\partial x^5} = 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 136x^4y^2 + 240x^2y^4 + 120y^6.$$

Recursive computation of Taylor coefficients.

Example: $y' = x^2 + y^2$

Set $y(x_0+h) = y_0 + hy_1 + h^2y_2 + \ldots,\ \ x^2 = x_0^2 + 2x_0h + h^2$,

$$
\begin{aligned}
\text{develop:}\quad y' = &\quad y_1 \quad + 2y_2h \quad + 3y_3h^2 \quad + 4y_4h^3 + \ldots \\
= &\quad x_0^2 \quad + 2x_0h \qquad\quad + h^2 \\
&+ y_0^2 + 2y_1y_0h + 2y_0y_2h^2 + 2y_0y_3h^3 \\
&\qquad\qquad\qquad\qquad y_1^2h^2 + 2y_1y_2h^3 + \ldots
\end{aligned}
$$

$$\Rightarrow\ \ y_1 = x_0^2 + y_0^2,\ \ 2y_2 = 2x_0 + 2y_1y_0,\ \ 3y_3 = 1 + 2y_0y_2 + y_1^2, \ldots$$

Recursive computation of Taylor coefficients.

Example: $y' = x^2 + y^2$

Set $y(x_0+h) = y_0 + hy_1 + h^2y_2 + \dots,\quad x^2 = x_0^2 + 2x_0h + h^2,$

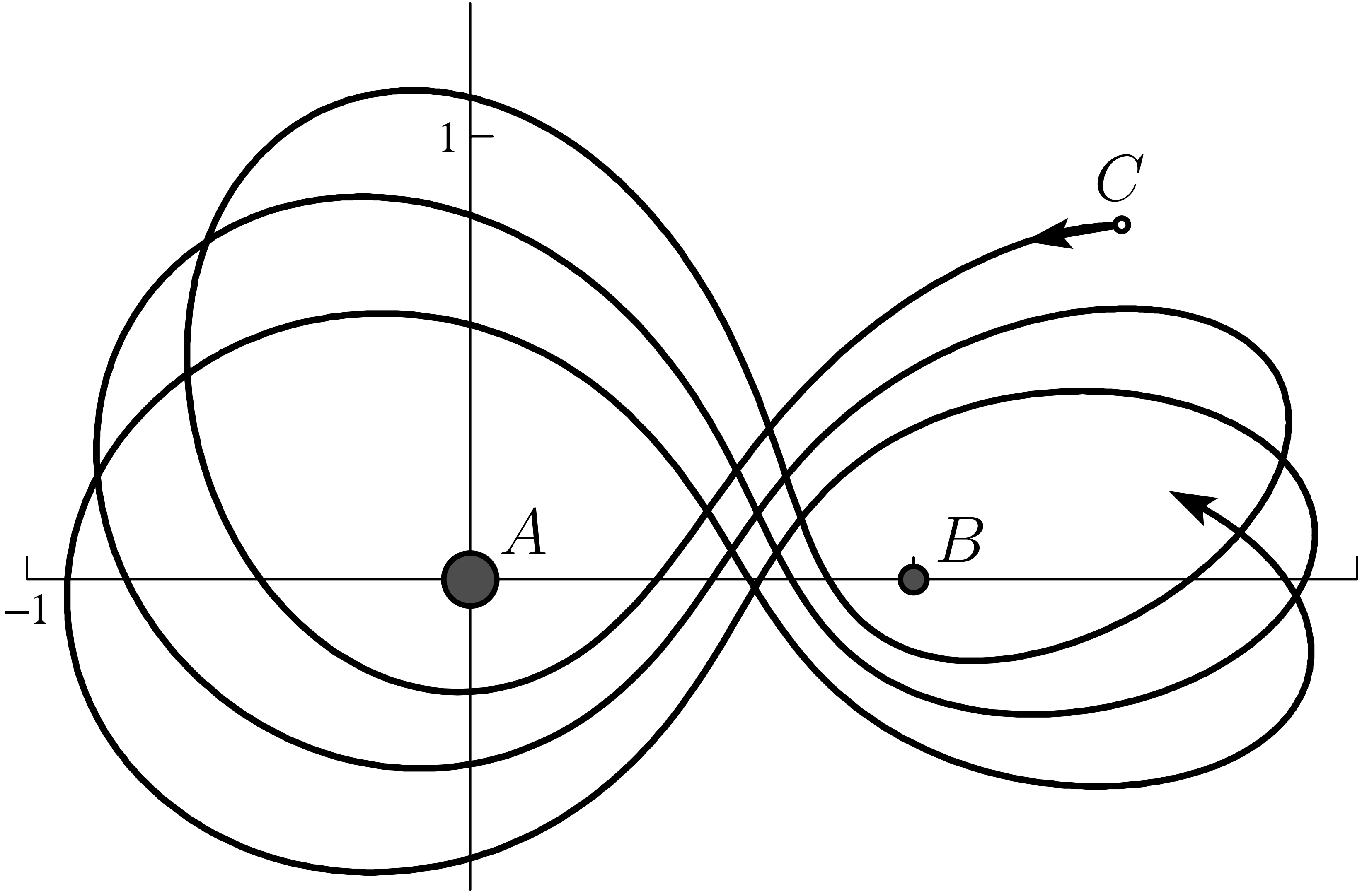
$$\begin{array}{llllll} \text{develop:}\quad y' = & y_1 & +2y_2h & +3y_3h^2 & +4y_4h^3 & +\dots \\ \qquad\qquad = & x_0^2 & +2x_0h & +h^2 & & \\ & +y_0^2 & +2y_1y_0h & +2y_0y_2h^2 & +2y_0y_3h^3 & \\ & & & y_1^2h^2 & +2y_1y_2h^3 & +\dots \end{array}$$

$$\Rightarrow \quad y_1 = x_0^2 + y_0^2,\ \ 2y_2 = 2x_0 + 2y_1y_0,\ \ 3y_3 = 1 + 2y_0y_2 + y_1^2, \dots$$

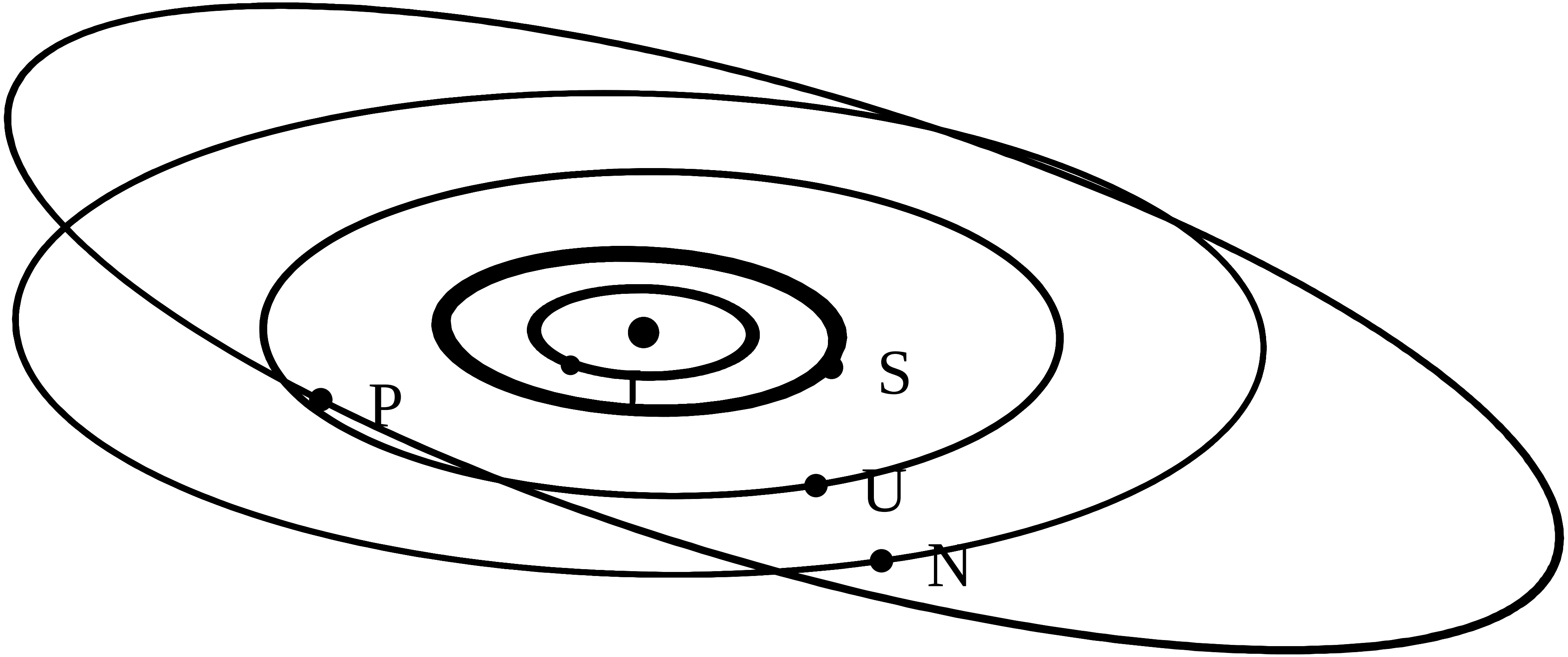
$$\alpha = aa + bb,\ 2\beta = 2\alpha b + 2a,\ 3\gamma = 2\beta b + \alpha\alpha + 1,$$
$$4\delta = 2\gamma b + 2\alpha\beta,\ 5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta$$
$$6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma,\ \text{etc.}$$

Surprise: Euler invented this too !! (E342), §663.

Example of two fixed centers was calculated by this method:



Another Example: The Outer Solar System (2000000 days).



General Principle of Movement (E112, E177).

"While physicists call these "Newton's equations", they occur nowhere in the work of Newton or of anyone else prior to 1747."

"The discovery of this principle seems so easy, from Newtonian ideas, that is has never been attributed to anyone but Newton; such is the universal ignorance of the true history of mechanics."

"The structure ... we owe to many great mathematicians; among the founders, Newton is joined by Huygens, Leibniz, and James Bernoulli, and the great architect is Euler."

(C. Truesdell, *Essays in the History of Mechanics,* 1968)

$$\text{I. } \frac{2ddx}{dt^2} = \frac{X}{M}; \quad \text{II. } \frac{2ddy}{dt^2} = \frac{Y}{M}; \quad \text{III. } \frac{2ddz}{dt^2} = \frac{Z}{M}$$

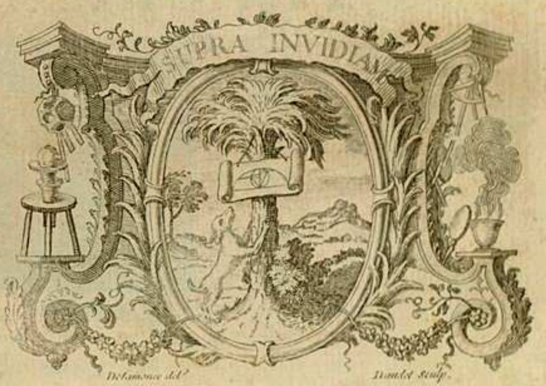
MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET
MATHESEOS SVBLIMIORIS PROFESSORE.

TOMVS I.

INSTAR SVPPLEMENTI AD COMMENTAR.
ACAD. SCIENT. IMPER.

PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.

METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*



LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

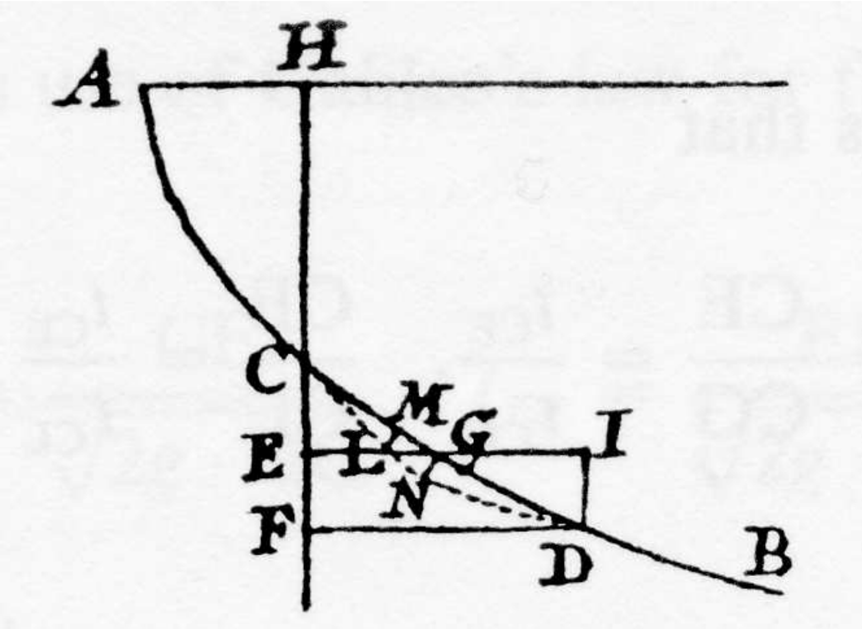
Mechanica 1736 E15

Variational Calc. 1744 E65

Variational Calculus.

Challenge of Joh. Bern. (1696)

$$\int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \text{min!}$$

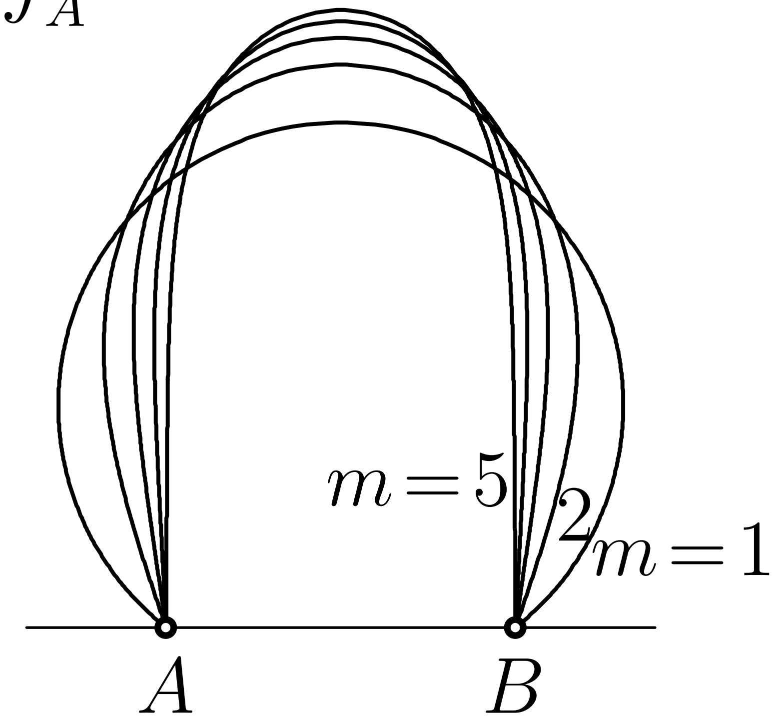


Jakob could solve.

Challenge of Jak. Bern. (1697)

$$\int_A^B y^m \, dx = \text{max!}$$

$$\int_A^B ds = L \quad \text{(given)}$$



Joh. could not solve.

Euler (1744):

E65: Methodus inveniendi lineas curvas maximi minimive ...
The break-through. Euler's general equation in extreme clarity, 66 (sixtysix) fully solved examples !!

"... eines der schönsten mathematischen Werke, die je geschrieben worden sind" (C. Carathéodory)

Problem. Find $y(x)$ such that

$$J = \int_a^b Z\,dx = \text{min! vel max!} \quad \text{where } Z = f(x, y, p), \; p = \frac{dy}{dx}\,.$$

Euler (1744):

E65: Methodus inveniendi lineas curvas maximi minimive ...
The break-through. Euler's general equation in extreme clarity, 66 (sixtysix) fully solved examples !!

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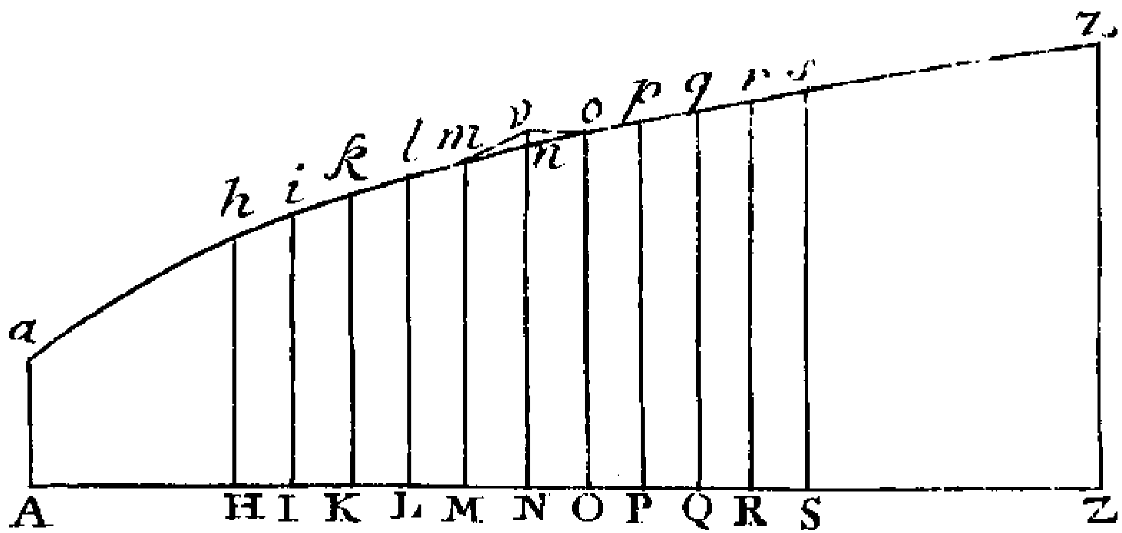
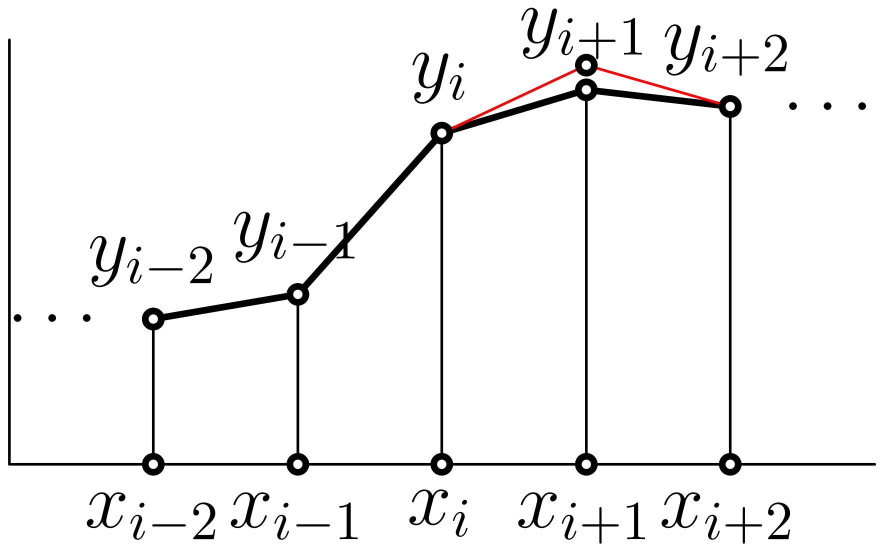
Euler's Solution.





$$p_i = \frac{y_{i+1} - y_i}{dx},$$

$$Z_i = f(x_i, y_i, p_i), \qquad J = (\ldots + Z_{i-1} + Z_i + Z_{i+1} + Z_{i+2} + \ldots)\, dx$$

Then set

$$\frac{\partial f}{\partial y} = N, \;\; \frac{\partial f}{\partial p} = P, \;\; \frac{\partial Z_i}{\partial y_{i+1}} = \frac{1}{dx} P_i, \;\; \frac{\partial Z_{i+1}}{\partial y_{i+1}} = -\frac{1}{dx} P_{i+1} + N_{i+1}\,.$$

With this, the condition of optimality is, for all i,

$$\frac{\partial J}{\partial y_{i+1}} = N_{i+1} - \frac{P_{i+1} - P_i}{dx} = 0 \;\text{ or }\quad N - \frac{dP}{dx} = 0\;!!$$

A First Integration.

in our notation Euler's equation is

$$\frac{\partial Z}{\partial y} - \frac{d}{dx}\left(\frac{\partial Z}{\partial p}\right) = 0 \quad \text{or} \quad \frac{\partial Z}{\partial y} - \frac{\partial^2 Z}{\partial x\,\partial p} - \frac{\partial^2 Z}{\partial y\,\partial p}\cdot y' - \frac{\partial^2 Z}{\partial p\,\partial p}\cdot y'' = 0\,.$$

If Z is independent of x, can be replaced by

$$Z - p\cdot\frac{\partial Z}{\partial p} = C \qquad C = \text{Const.}$$

Example: the Brachystochrone.

$$\int_a^b \frac{\sqrt{1+p^2}}{\sqrt{y}}\,dx = \text{min!} \quad \Rightarrow \quad \frac{\sqrt{1+p^2}}{\sqrt{y}} - \frac{p^2}{\sqrt{y}\sqrt{1+p^2}} = C$$

$$\text{or} \quad -1 = \sqrt{1+p^2}\sqrt{y}\cdot C$$

same equation as above.

... and another young genius ...

challenges the (now) most famous mathematician: 19 years old *Ludovico de la Grange Tournier* writes 1755 a letter to *Vir amplissime atque celeberrime L. Euler*. But this time, *Vir praestantissime atque excellentissime Lagrange* receives a kind and enthusiastic answer.

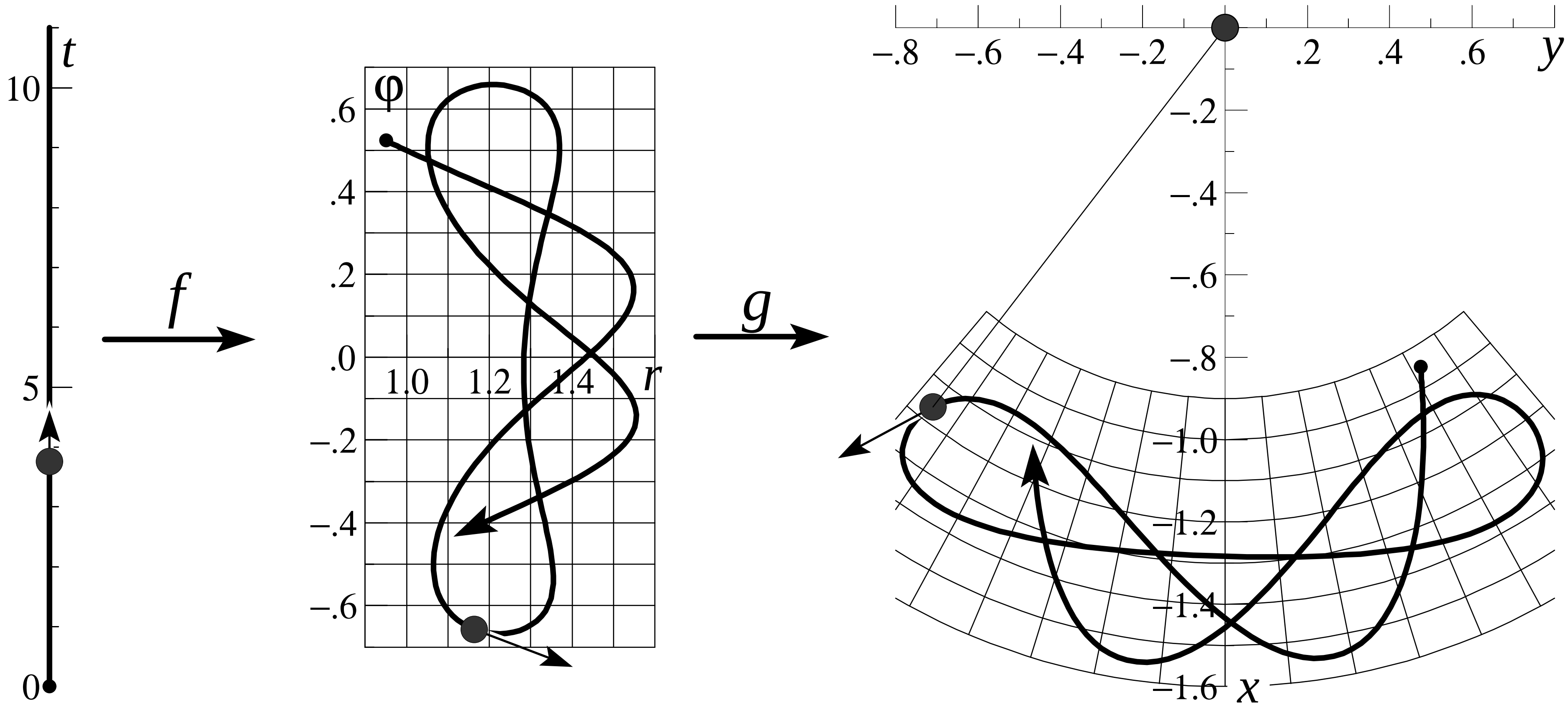
- Lagrange simplifies the derivation of Euler's equations by the Variational Calculus notation δy ;
- Lagrange introduces his multiplyers for constrained optimization ;
- and establishes the relation between Euler's equations

$$\text{I. } \frac{2ddx}{dt^2} = \frac{X}{M}; \quad \text{II. } \frac{2ddy}{dt^2} = \frac{Y}{M}; \quad \text{III. } \frac{2ddz}{dt^2} = \frac{Z}{M} \qquad \Leftrightarrow \quad N - \frac{dP}{dx} = 0$$

for the variational problem $\int (T - U)\, dt =$min!, $\Rightarrow$ Lagrangian mechanics (1788).

Example An Elastic Pendulum.

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2), \quad U = r \cdot \cos\varphi + \lambda\,\frac{(r-1)^2}{2}, \quad L = T - U$$



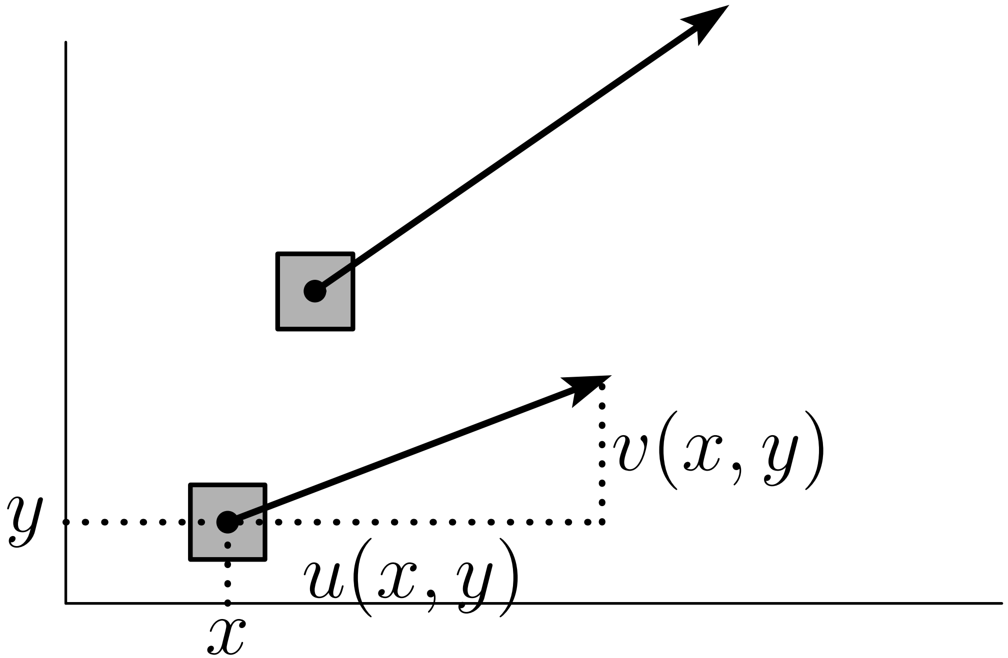
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \qquad \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 0$$

represent differential equations of movement.

Hydrodynamics (E226, E227, 1753)

"In the cosmos of *hydrodynamics*, Newton is the Ouranos, Daniel Bernoulli the Kronos, and Euler the Zeus. (. . .) Book II of Newtons *Principia*, nearly a third of the whole, concerns fluids. Almost all of the results are original, and but few correct."

(C. Truesdell, *Prologue* in Euler's *Opera Omnia*, ser. II, vol. 12)



Idea: represent fluid by

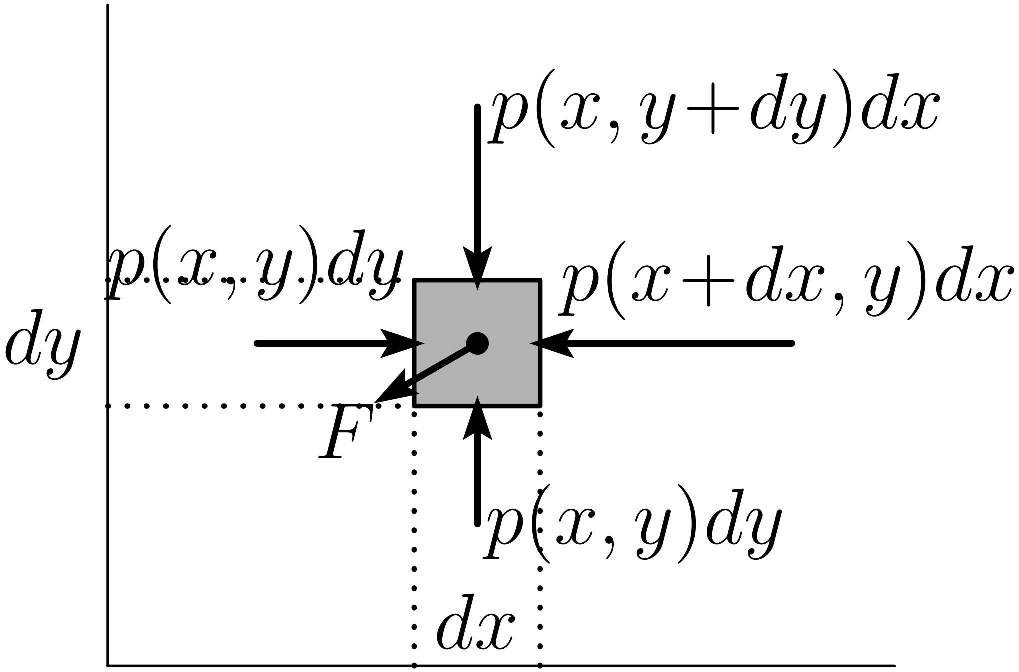
- pressure $p(x, y, t)$
- speed in x-dir. $u(x, y, t)$
- speed in y-dir. $v(x, y, t)$

Acceleration of a piece of water has three sources:

- changement of speed in time $\dfrac{\partial u}{\partial t}, \quad \dfrac{\partial v}{\partial t}$

- force due to

variation of pressure $p(x, y, t)$

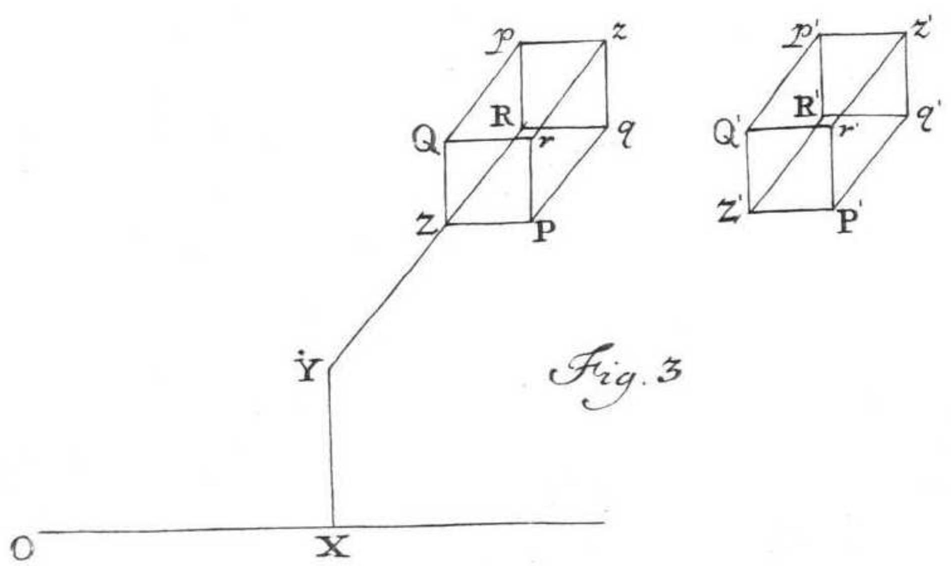


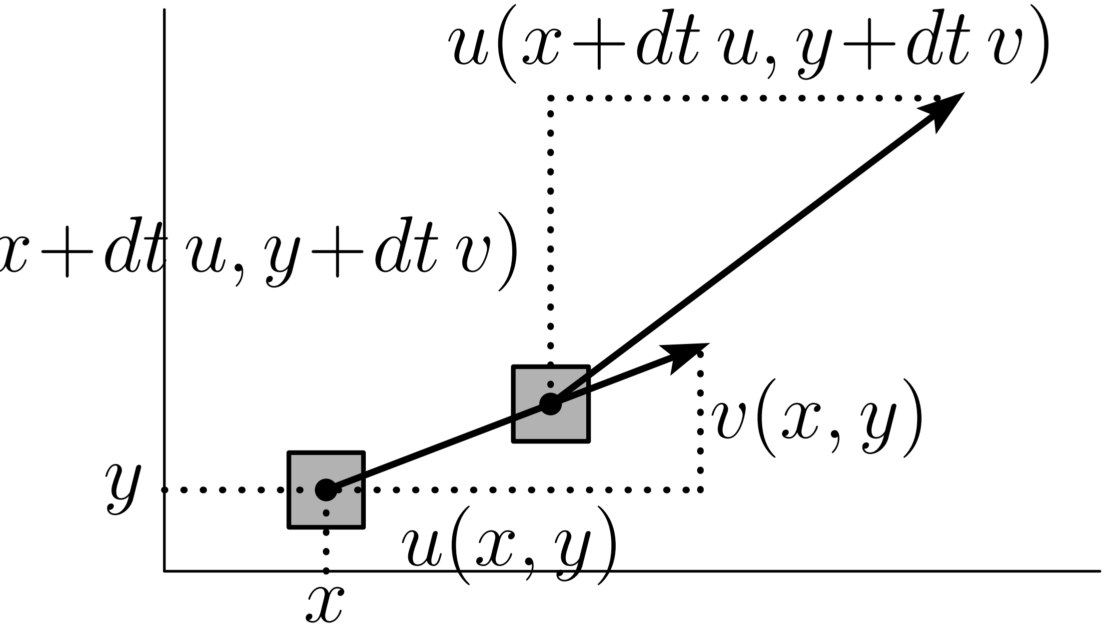
$$\frac{F}{M} =$$

$$\begin{pmatrix} p(x,y) - p(x+dx,y)dy \\ p(x,y) - p(x,y+dy)dx \end{pmatrix} \frac{1}{M}$$

$$= \begin{pmatrix} -\frac{\partial p}{\partial x} \\ -\frac{\partial p}{\partial y} \end{pmatrix} \frac{dx\, dy}{M} = \begin{pmatrix} -\frac{\partial p}{\partial x} \\ -\frac{\partial p}{\partial y} \end{pmatrix} \frac{1}{\rho}$$

- force due to displacement (Euler E226, E227, 1753)





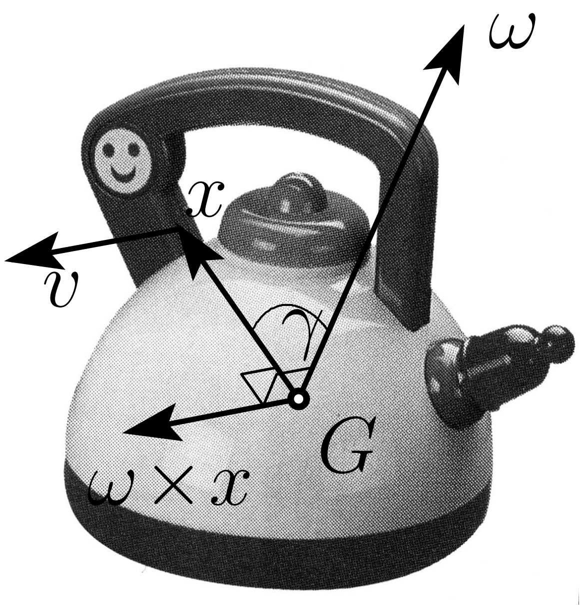
$$\begin{pmatrix} du \\ dv \end{pmatrix} =$$

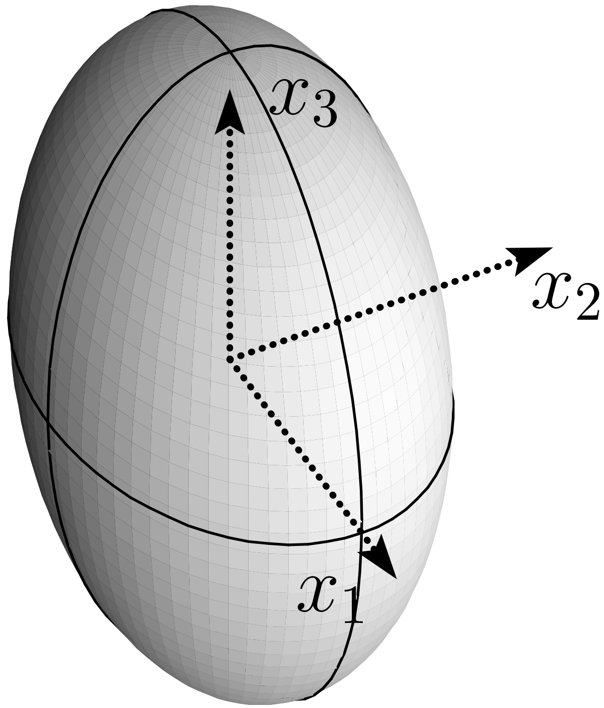
$$\begin{pmatrix} u(x+u\,dt, y+v\,dt) - u(x,y) \\ v(x+u\,dt, y+v\,dt) - v(x,y) \end{pmatrix}$$

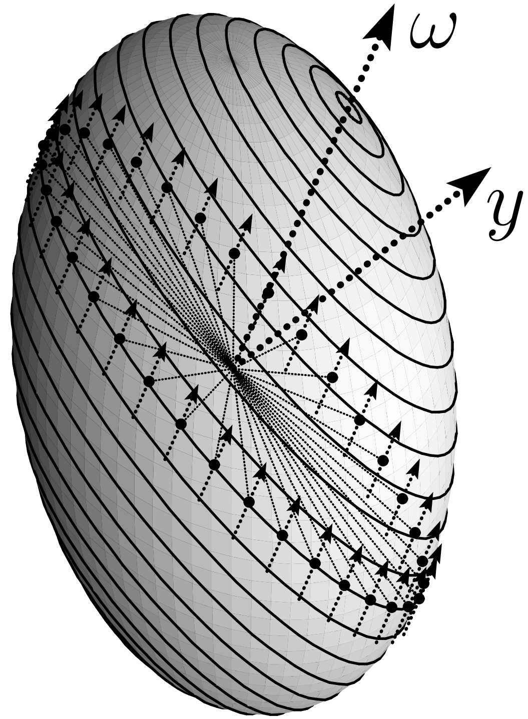
$$= \begin{pmatrix} u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} \\ u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} \end{pmatrix} dt$$

“Nous n’avons donc qu’à égaler
ces forces accélératrices
avec les accélérations actuelles.”

Rigid Body Dynamics (E292, 1758)





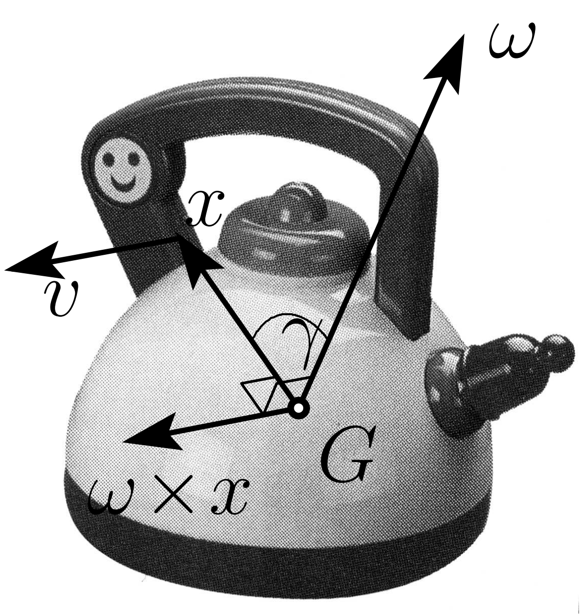


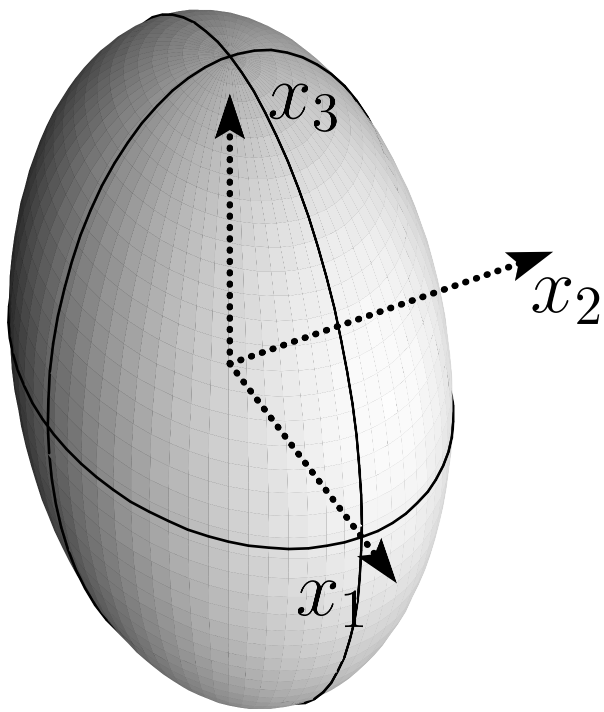
Kinetic Energy: (E291)

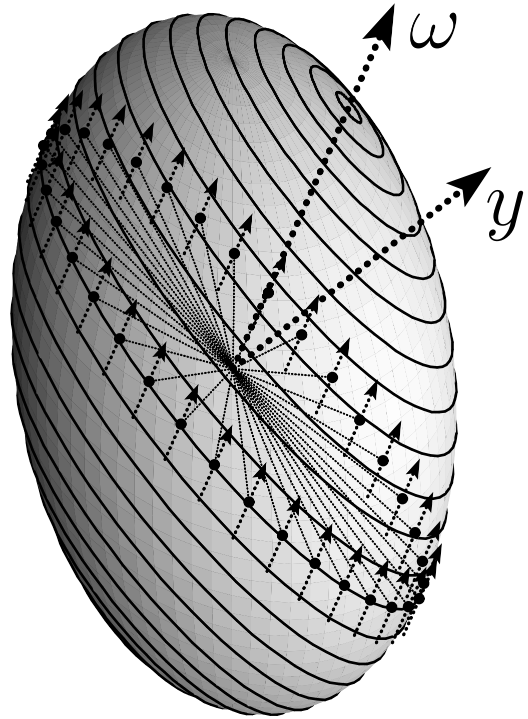
$$v = \omega \times x = \begin{pmatrix} \omega_2 x_3 - \omega_3 x_2 \\ \omega_3 x_1 - \omega_1 x_3 \\ \omega_1 x_2 - \omega_2 x_1 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$T = \frac{1}{2} \int_{\mathcal{B}} \|\omega \times x\|^2 \, dm = \frac{1}{2} \omega^T \Theta \omega = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \, .$$

Rigid Body Dynamics (E292, 1758)







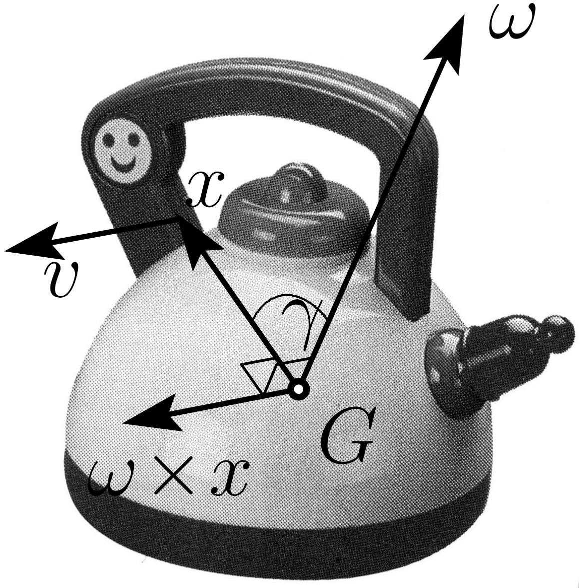
Angular Momentum:

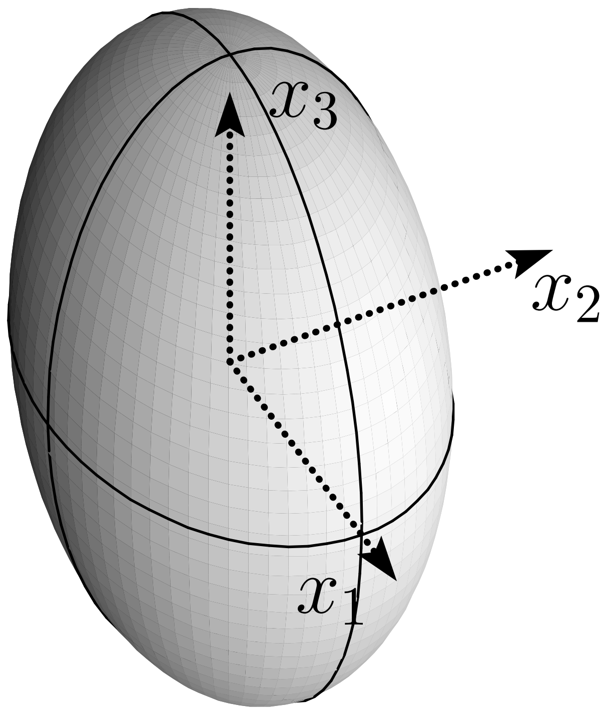
$$y = \int_{\mathcal{B}} (x \times v)\, dm = \int_{\mathcal{B}} (x \times (\omega \times x))\, dm \;= \Theta\, \omega,$$

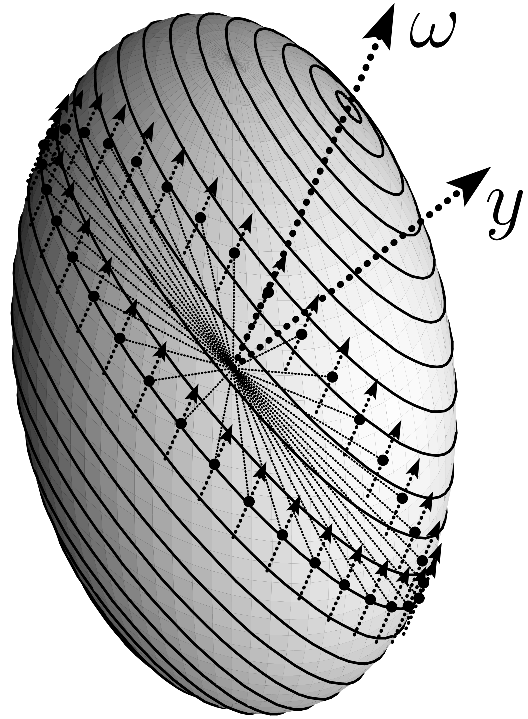
or, in the principal axes coordinates,

$$y_k = I_k\, \omega_k\,.$$

E479 (1775). New principle: Without forces, angular Momentum y constant vector in space







In absence of exterior forces, the angular momentum, seen from the fixed space, is a constant vector. Thus, seen from the body's coordinates, y rotates around ω in the other direction i.e.,

$$v = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Big\| \quad \begin{pmatrix} \dot y_1 \\ \dot y_2 \\ \dot y_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

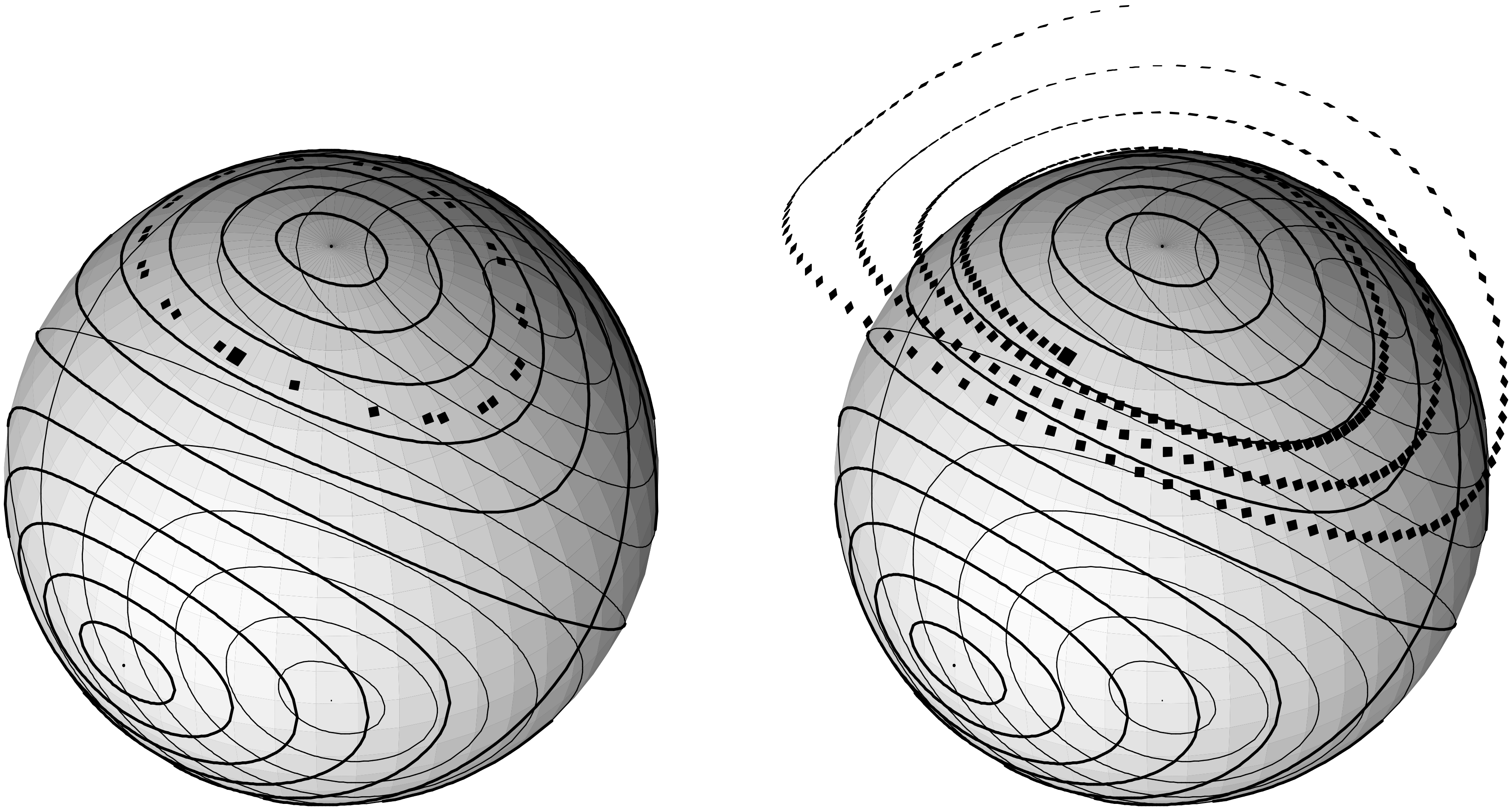
Inserting $y_k = I_k \omega_k$, i.e., $\omega_k = y_k / I_k$, this gives

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & y_3/I_3 & -y_2/I_2 \\ -y_3/I_3 & 0 & y_1/I_1 \\ y_2/I_2 & -y_1/I_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} (I_3^{-1} - I_2^{-1})\, y_3 y_2 \\ (I_1^{-1} - I_3^{-1})\, y_1 y_3 \\ (I_2^{-1} - I_1^{-1})\, y_2 y_1 \end{pmatrix}$$

or

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \begin{pmatrix} y_1/I_1 \\ y_2/I_2 \\ y_3/I_3 \end{pmatrix},$$

two ways of writing the Euler equations as a Poisson system, once with the sphere, once with an ellipsoid, as first integral, whose intersections determine the solution curves.



$$dx + \frac{cc - bb}{aa} \cdot yz\,dt = \frac{2gP\,dt}{Maa}$$

$$dy + \frac{aa - cc}{bb} \cdot xz\,dt = \frac{2gQ\,dt}{Mbb}$$

$$dz + \frac{bb - aa}{cc} \cdot xy\,dt = \frac{2gR\,dt}{Mcc}$$

250 years of Euler’s Rigid Body Equations!!

Quaternion multiplication.

Numerical software for rigid body computations use quaternion operations, often called Euler parameters.

Do quaternions really appear in Euler's work ??

$$\begin{pmatrix} e_0 & -e_1 & -e_2 & -e_3 \\ e_1 & e_0 & -e_3 & e_2 \\ e_2 & e_3 & e_0 & -e_1 \\ e_3 & -e_2 & e_1 & e_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

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Quaternion multipl.

$$\begin{aligned} A &= a\alpha + b\beta + c\gamma + d\delta \\ B &= a\beta - b\alpha - c\delta + d\gamma \\ C &= a\gamma + b\delta - c\alpha - d\beta \\ D &= a\delta - b\gamma + c\beta - d\alpha \end{aligned}$$

Probl. 4 squares. (E445)

Yes, they do !! In Euler's proof of a theorem in Number Theory, that every number is a sum of four squares, a problem going back to Diophantus ...

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$$\begin{pmatrix} e_0 & -e_1 & -e_2 & -e_3 \\ e_1 & e_0 & -e_3 & e_2 \\ e_2 & e_3 & e_0 & -e_1 \\ e_3 & -e_2 & e_1 & e_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Quaternion multipl.

$$\begin{aligned} A &= \alpha a + b\beta + c\gamma + d\delta \\ B &= a\beta - b\alpha - c\delta + d\gamma \\ C &= a\gamma + b\delta - c\alpha - d\beta \\ D &= a\delta - b\gamma + c\beta - d\alpha \end{aligned}$$

Probl. 4 squares. (E445)

Yes, they do !! In Euler's proof of a theorem in Number Theory, that every number is a sum of four squares, a problem going back to Diophantus ...

... and with reference to "useless" research, we termine the conference ...

Thank you