Jakob Steiner

1796 ✳ 1863 †

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ne derange pas mes cercles !

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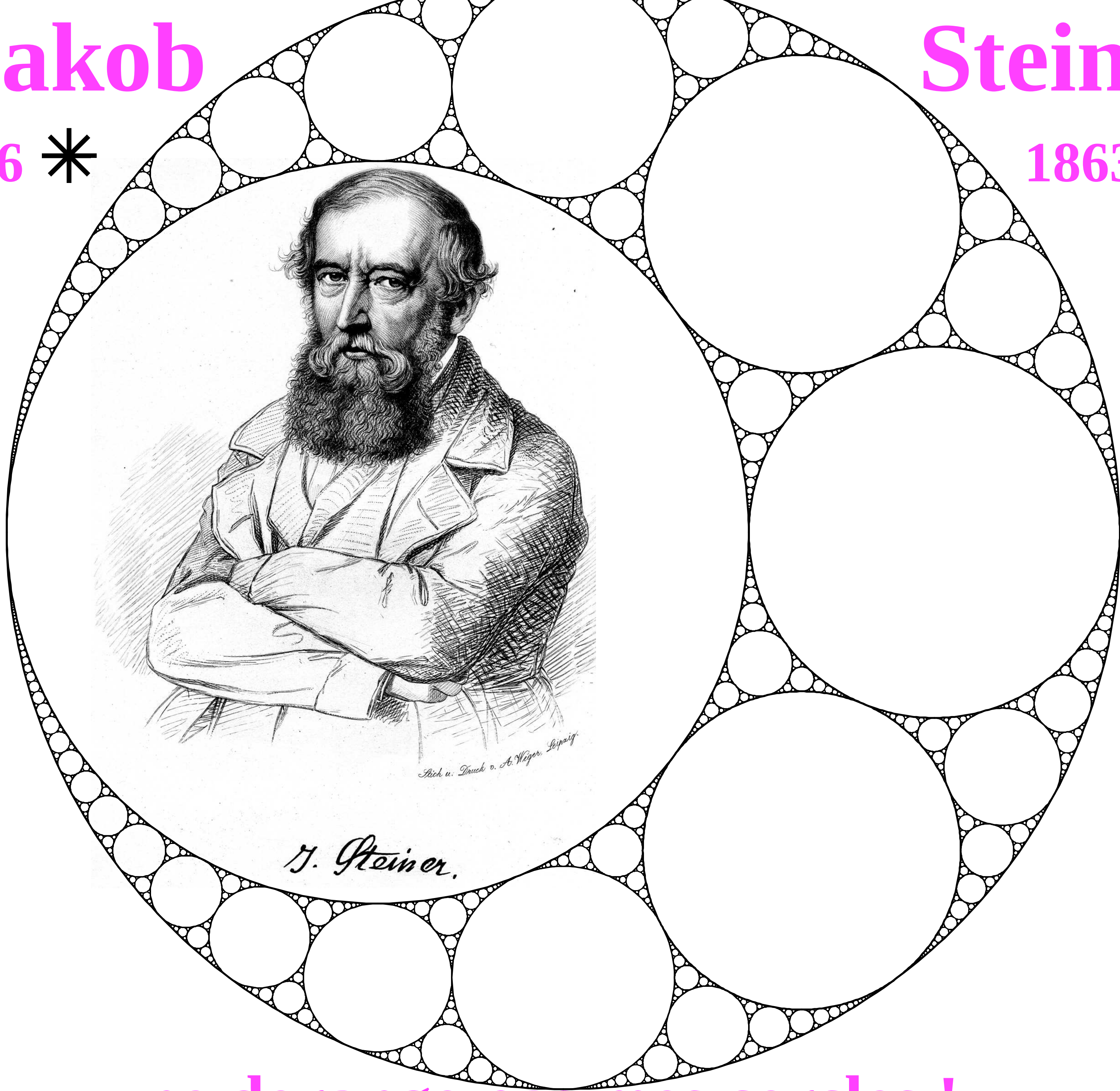
1796 ✳ **1863 †**

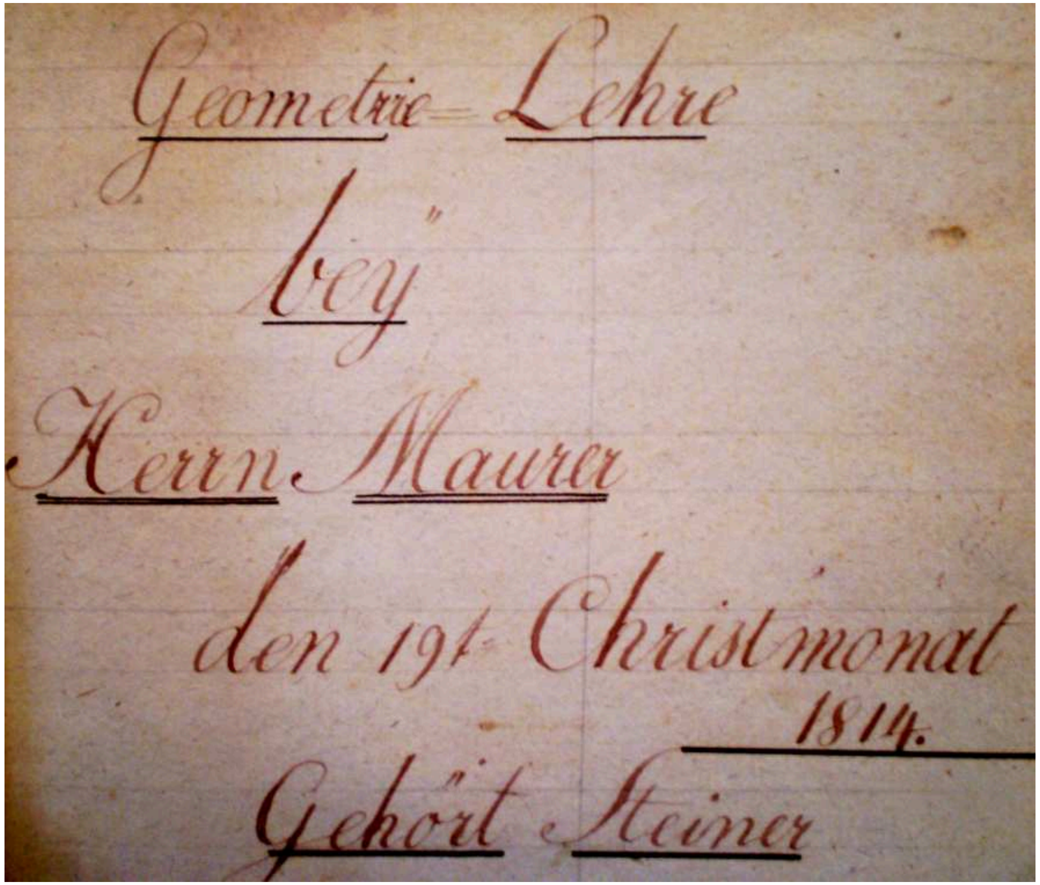


ne derange pas mes cercles !

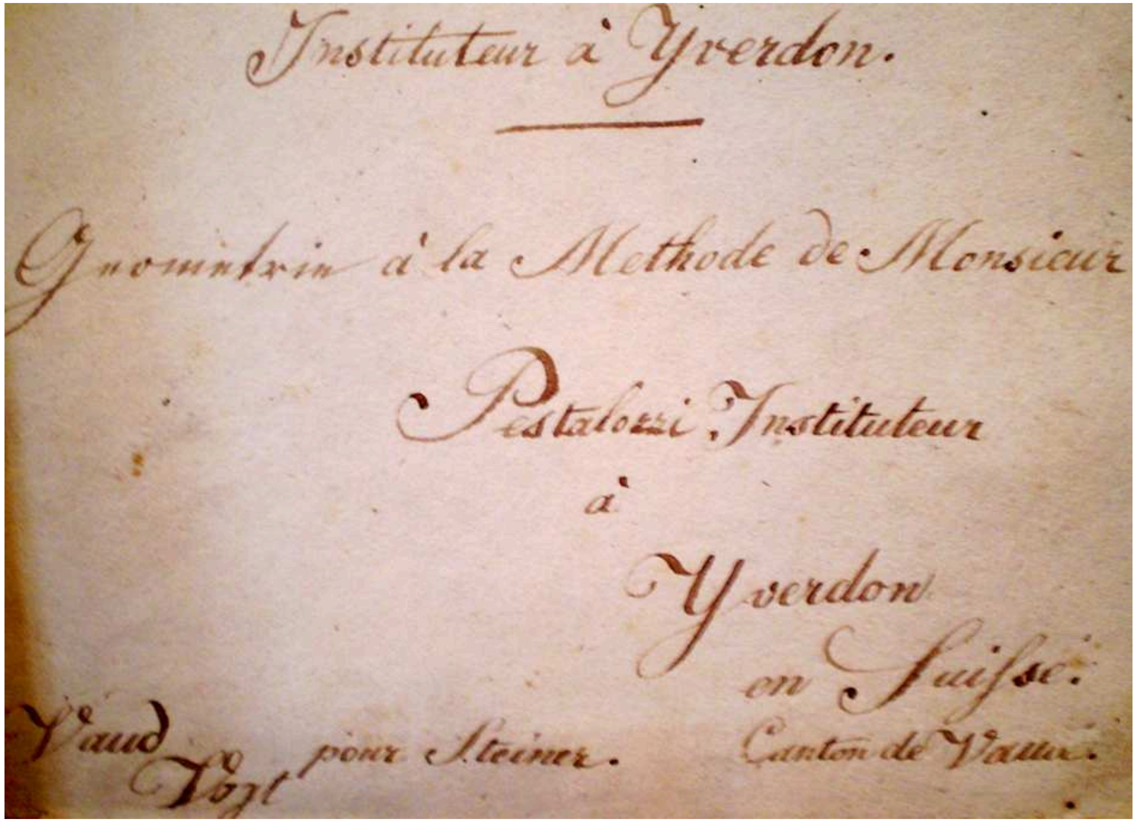
(Photos: Barbara Kummer)

born 1796
in Utzenstorf (Bern)
childhood: work at this farm,
no school education
until age of 18

1814 enters the Pestalozzi school in Yverdon ...



first as student ...



... then as “instituteur”.

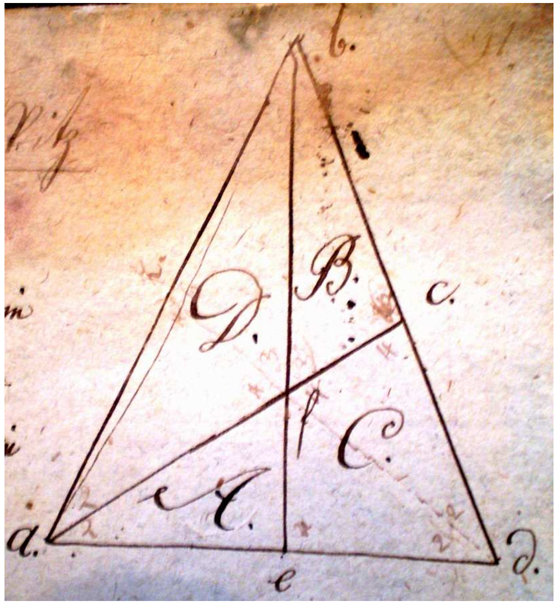
(Documents: Burgerbibliothek Bern)

With much energy: “Gefunden Samstag den 10. Christmonat 1814, $3 + 3 + 4$ St. daran gesucht, des Nachts um 1 Uhr gefunden.”

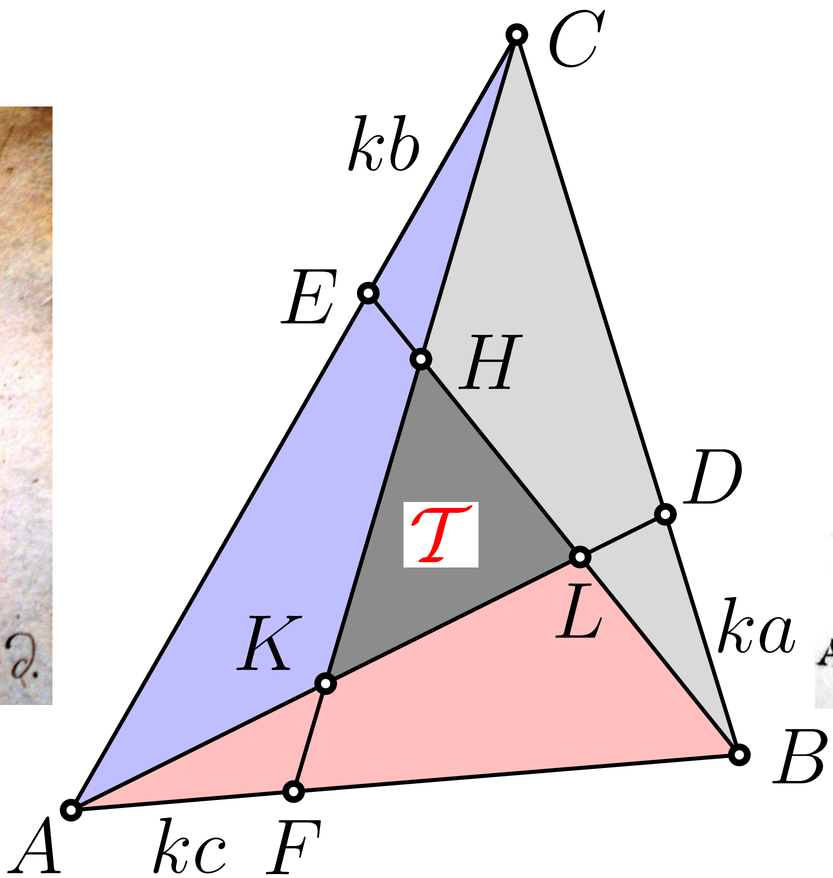
(Steiner, quoted from J.-P. Sydler, *L’Enseig. Math.* (1965), p. 241.)

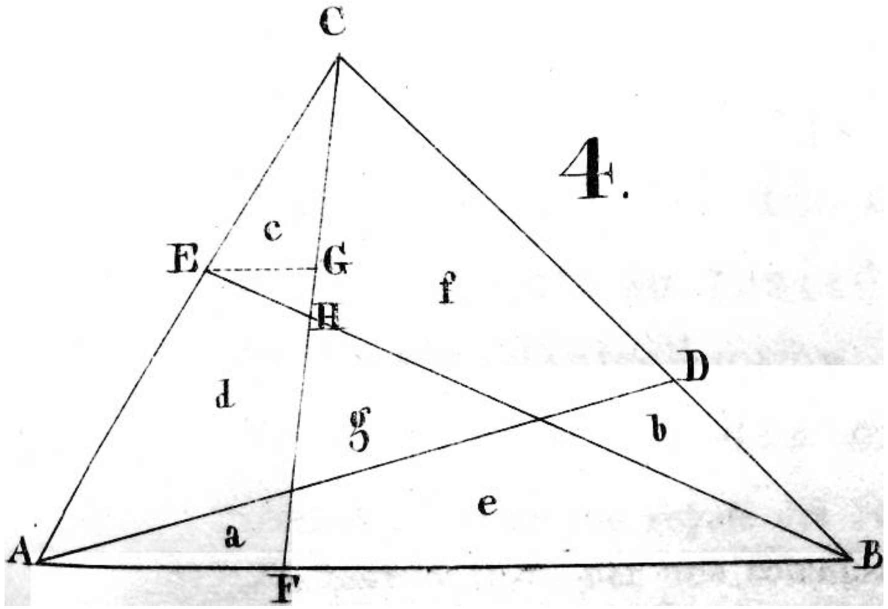
Life-long admiration for Pestalozzis non-dogmatical teaching ...

Example. Th. Clausen published in Crelle Journal **3**, No. 17 the determination of the area of $\mathcal{T}$ by complicated trigonometry.



(Steiner autogr. 1814)

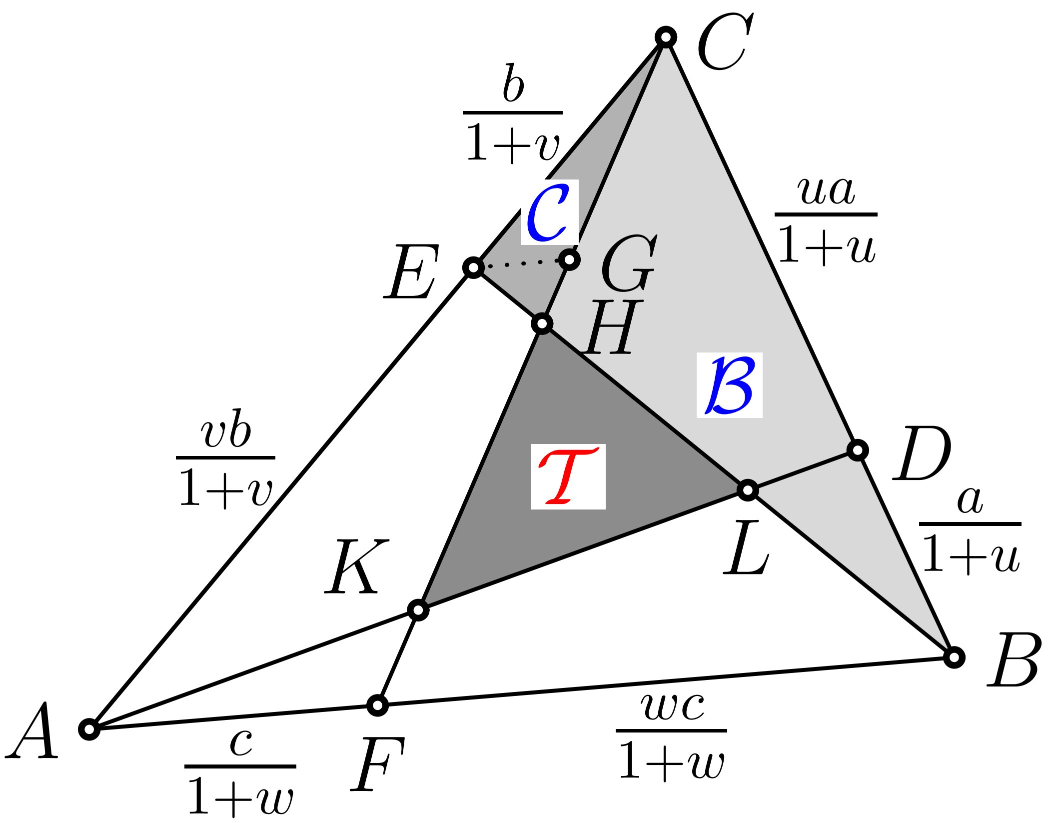




(Steiner, Crelle J. **3**, No. 19, 1828)

Three pages later (Steiner's answer): "Diese Aufgabe gehört zu einer Abtheilung von Elementar-Aufgaben, die die Pestalozzische Schule wegen mancherlei pädagogischer Vorzüge sehr in Betracht zog. Nach dieser Elementarübung wird die obige Aufgabe ohne die Hülfe trigonometrischer Functionen gelöset; auch kann sie allgemeiner gestellt werden, ..."

Steiner's solution: u, v, w ratios of side-lengths;



Draw EG parall. to AFB;

by Thales $EG : EC = AF : AC$

hence $EG = \frac{c}{(1+w)(1+v)}$;

By Eucl. I.41 and Thales

$\mathcal{C} : \mathcal{B} = EH : HB = EG : FB = \frac{1}{(1+v)w}$;

by Eucl. I.41 we have $\mathcal{C} + \mathcal{B} = \frac{1}{1+v} \cdot \mathcal{A}$;

can compute $\mathcal{B} = \frac{w}{1+w+wv} \cdot \mathcal{A}$;

cyclic permut. gives other triangles, subtract:

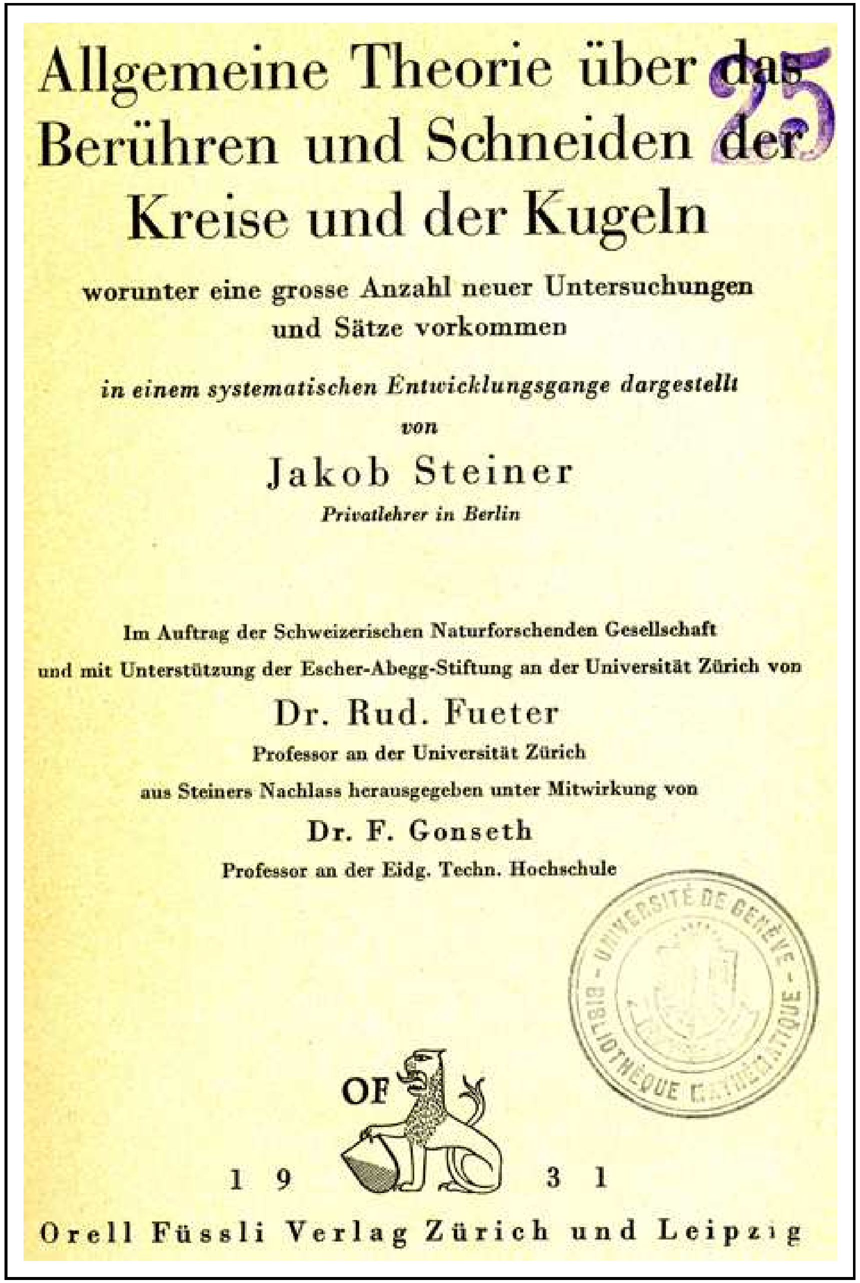
$$\mathcal{T} = \left(1 - \frac{u}{1+u+uw} - \frac{v}{1+v+vu} - \frac{w}{1+w+wv}\right) \cdot \mathcal{A}$$

$$= \frac{(uvw-1)^2}{(1+u+uw)(1+v+vu)(1+w+wv)} \cdot \mathcal{A}\,.$$

Observe: $\mathcal{T} = 0 \Leftrightarrow uvw = 1$ (Ceva's Theorem).

Result later forgotten and rediscovered as *Dudeney–Steinhaus theorem* or *Routh's theorem*.

1821: → Berlin, short career as teacher for lower maths in Werder Gymnasium, then "Privatlehrer" and starts writing a huge text on circle and sphere geometry ...



Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln

worunter eine grosse Anzahl neuer Untersuchungen und Sätze vorkommen

in einem systematischen Entwicklungsgange dargestellt

von

Jakob Steiner

Privatlehrer in Berlin

Im Auftrag der Schweizerischen Naturforschenden Gesellschaft und mit Unterstützung der Escher-Abegg-Stiftung an der Universität Zürich von

Dr. Rud. Fueter

Professor an der Universität Zürich

aus Steiners Nachlass herausgegeben unter Mitwirkung von

Dr. F. Gonseth

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OF

1931

Orell Füssli Verlag Zürich und Leipzig

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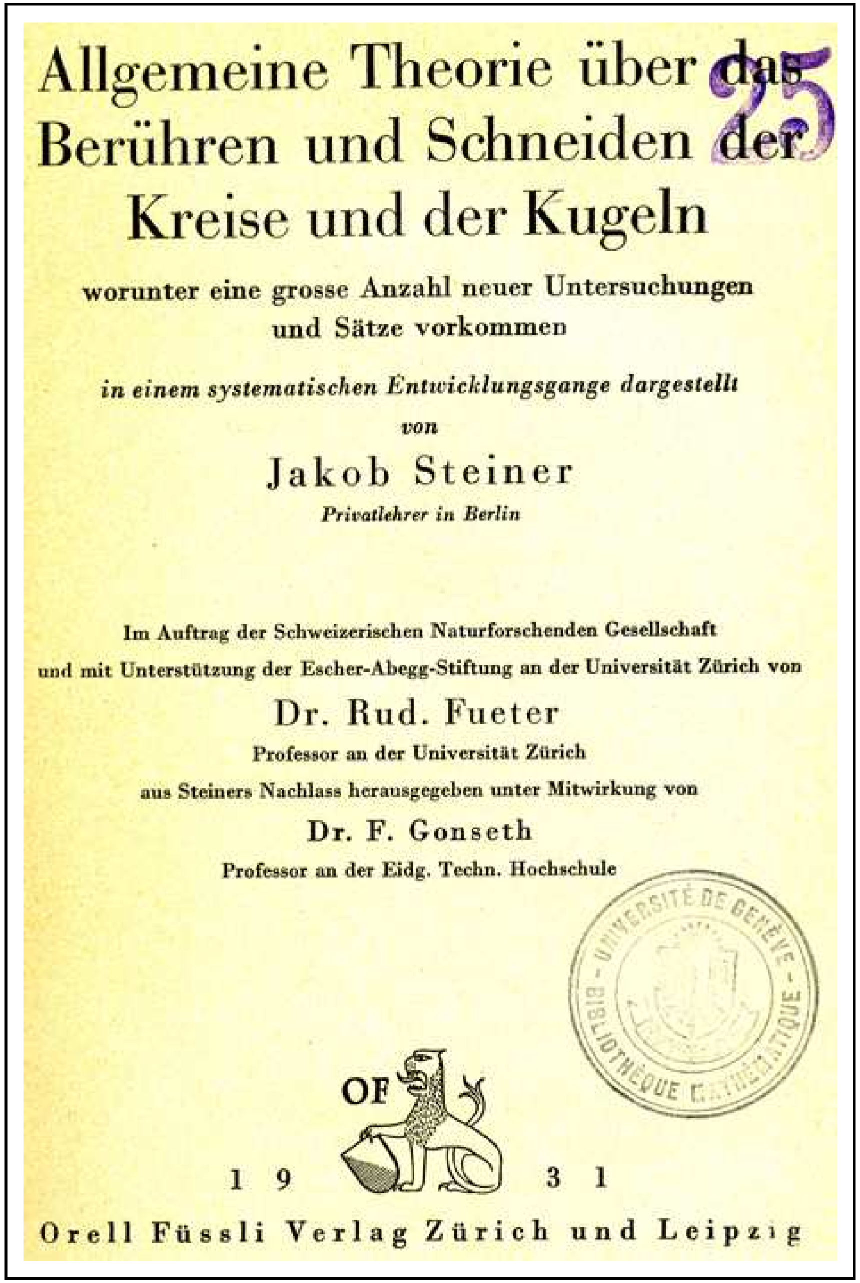

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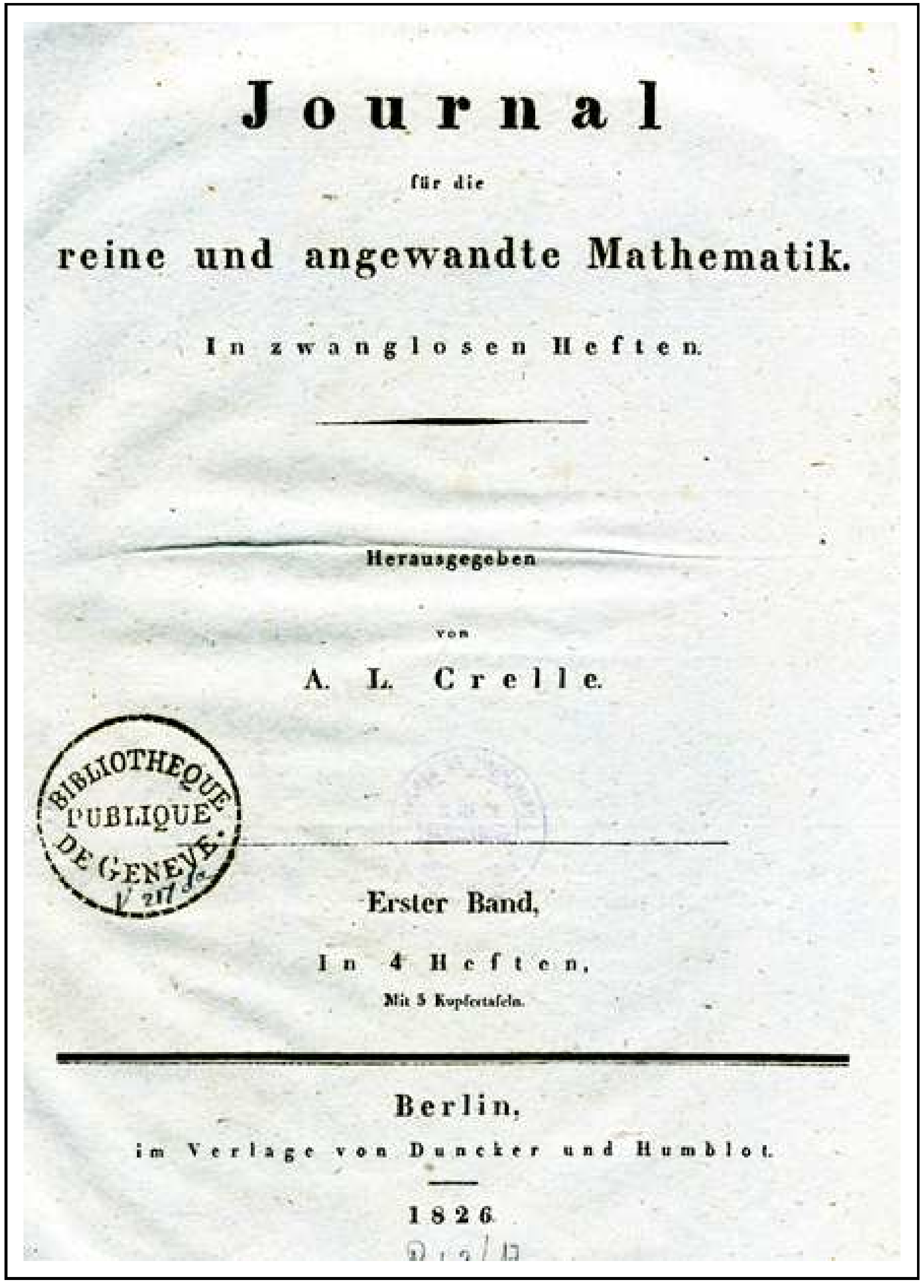


Journal

für die

reine und angewandte Mathematik.

In zwanglosen Heften.

Herausgegeben

von

A. L. Crelle.

Erster Band,

In 4 Heften,

Mit 5 Kupfertafeln.

Berlin,

im Verlage von Duncker und Humblot.

1826

... until he became a prolific contributor to the newly founded Crelle Journal.

Steiner's "Programm" as "Privatlehrer" in Berlin:

18.

Einige geometrische Betrachtungen.

(Von Herrn *Steiner.*)

Vor etwa drei Jahren sah sich der Verfasser dieser Abhandlung, zufälliger Weise, zur Beschäftigung mit der Aufgabe: 1) einen Kreis zu beschreiben, welcher drei andere, gegebene Kreise berührt; 2) mit der Malfattischen Aufgabe (14); so wie 3) mit dem XV. Theorem im IV. Buch der *Collect. mathem.* von Pappus; und 4) mit verschiedenen Porismen und der rein geometrischen Betrachtung der Curven und Flächen zweiten Grades, angeregt. Den Pappischen

(Steiner, Crelle J. **1**, p. 161, 1826)

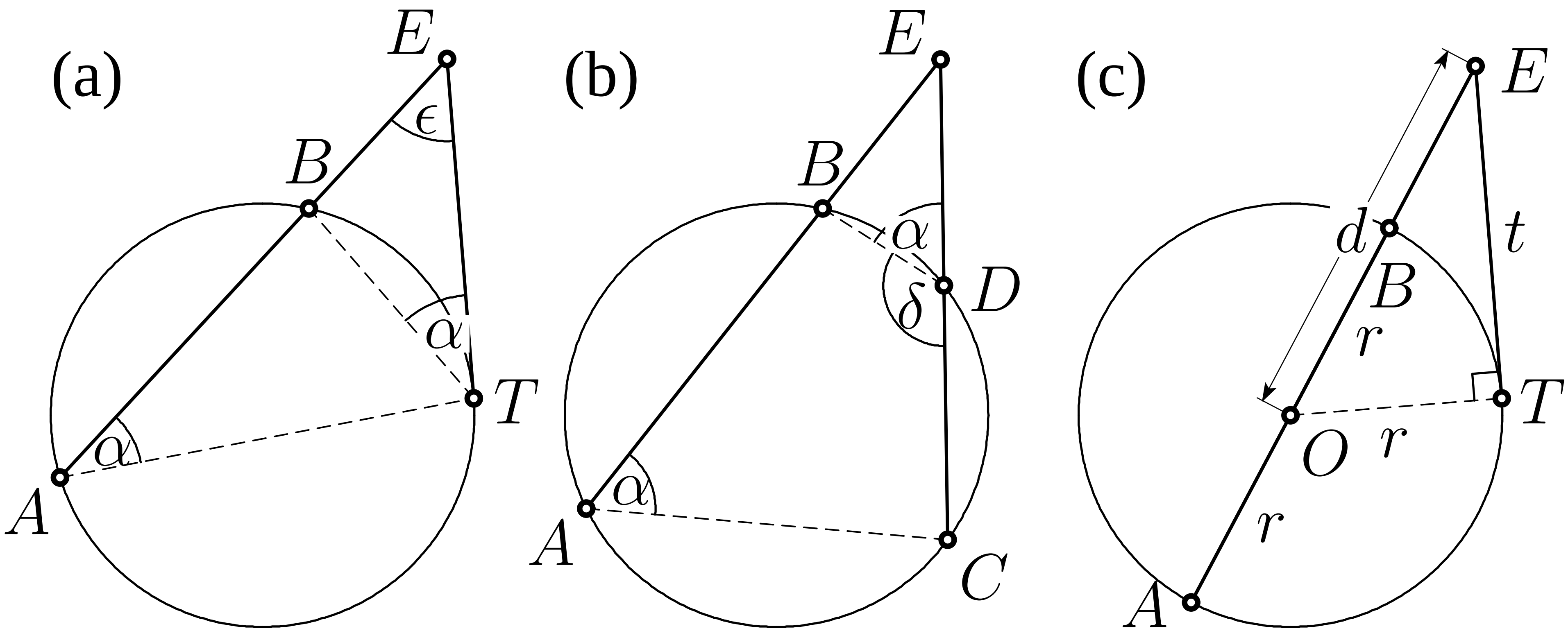
- Apollonius (Pappus VII);
- Malfatti;
- Pappus IV.15;
- conics.

"Das Bestreben des Verfassers ... den ... gemeinschaftlichen **Zusammenhang** zu finden". ⇒ General Theory of Circles.

Steiner's "Book of Lemmas":

1. Power of the point E with respect to the circle:

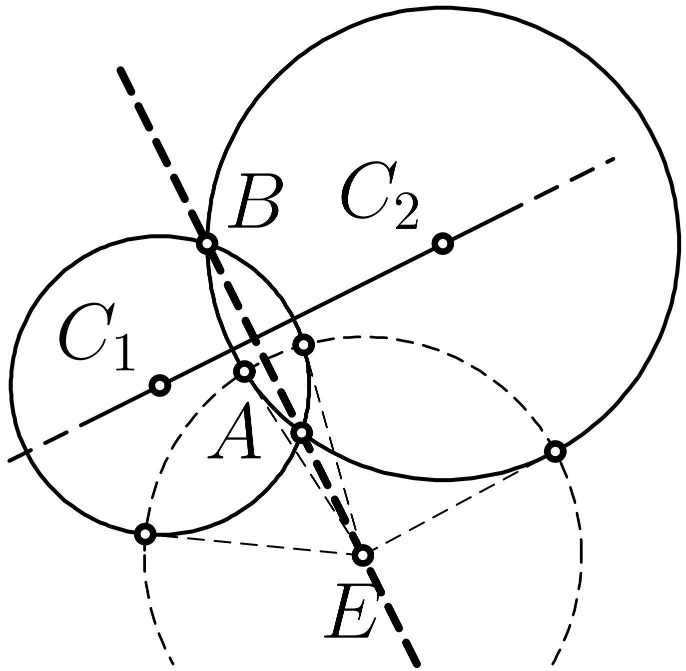
"In den Lehrbüchern der Geometrie findet man folgenden Satz bewiesen:"

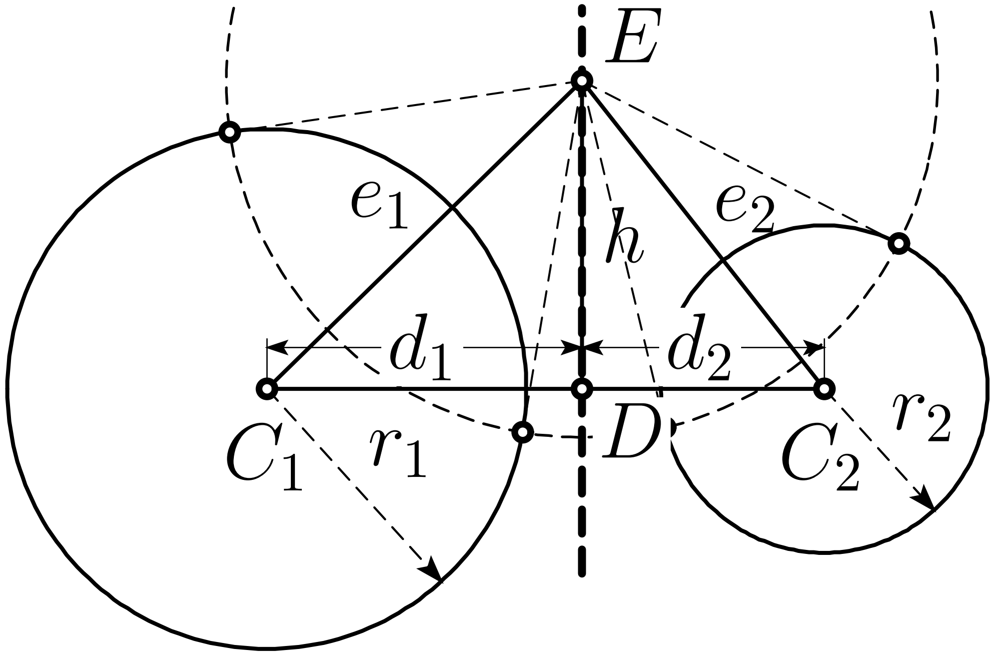


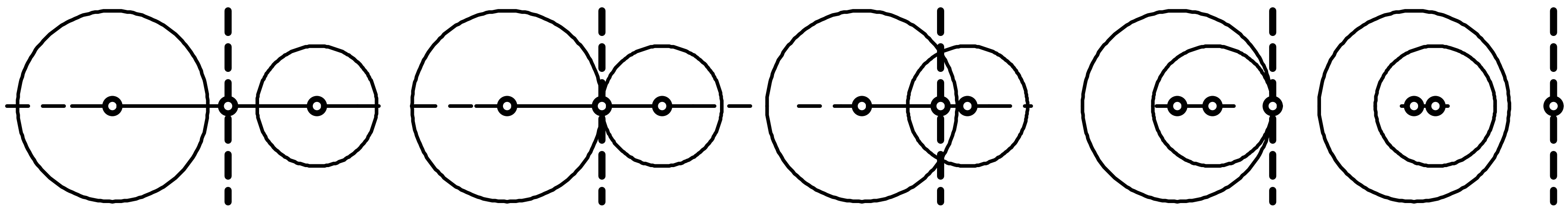
$$EB \cdot EA = ET^2 = ED \cdot EC = d^2 - r^2 = t^2 \qquad \text{(Eucl. III.36).}$$

"Dieses Product ... soll **'Potenz des Punkts in Bezug auf den Kreis'** heissen."

2. Line of equal powers for two circles:

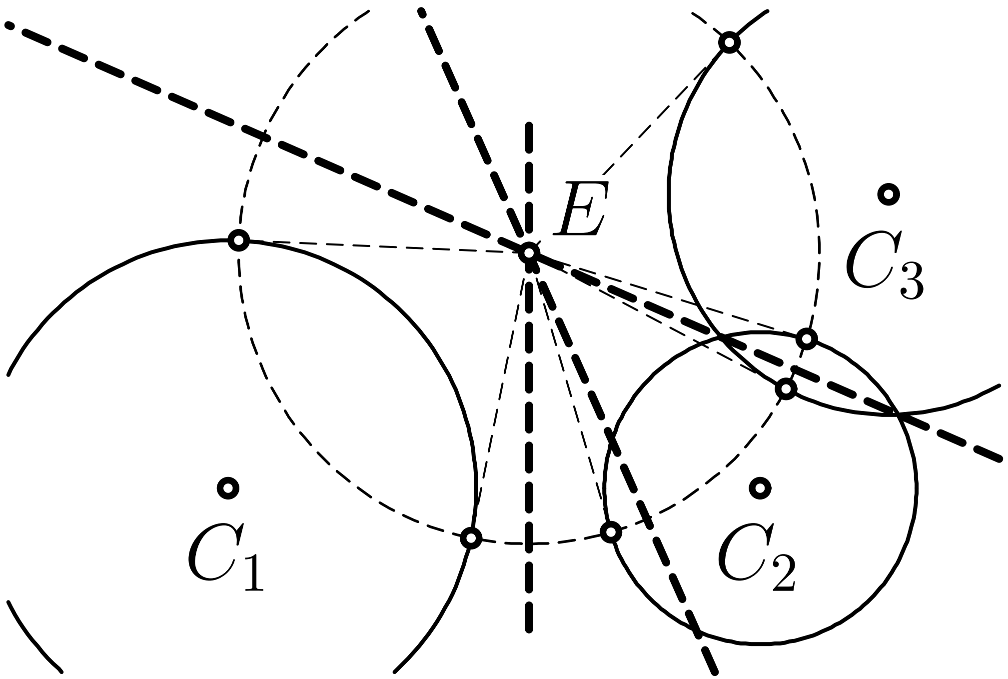


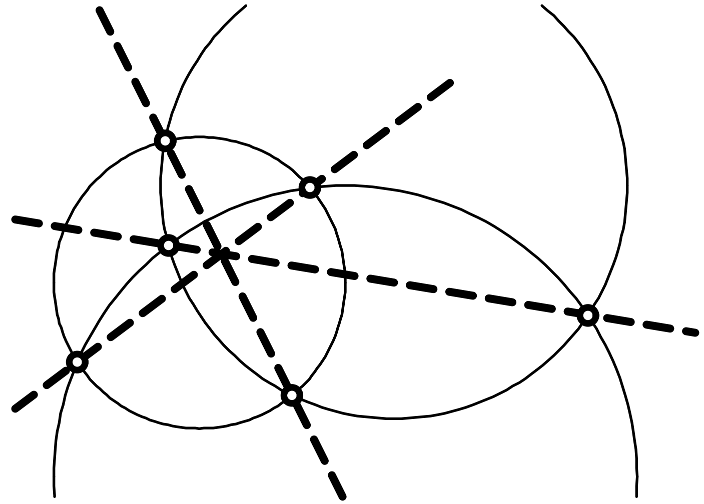




“Die Bedeutung der **gemeinschaftlichen** Potenz zweier Kreise, wovon schon bei Pappus und Vieta sich Spuren finden, lernte er durch ihre, von ihm gefundene vielseitige Anwendbarkeit erkennen”.

3. Point of equal powers for three circles:

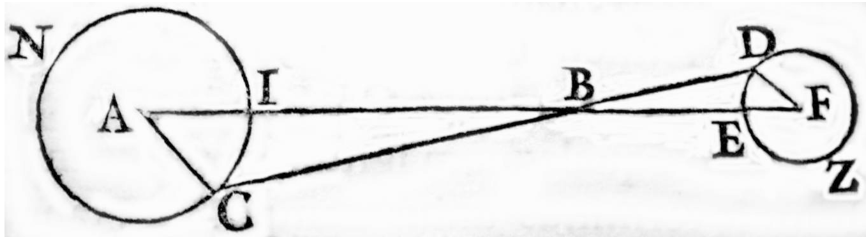




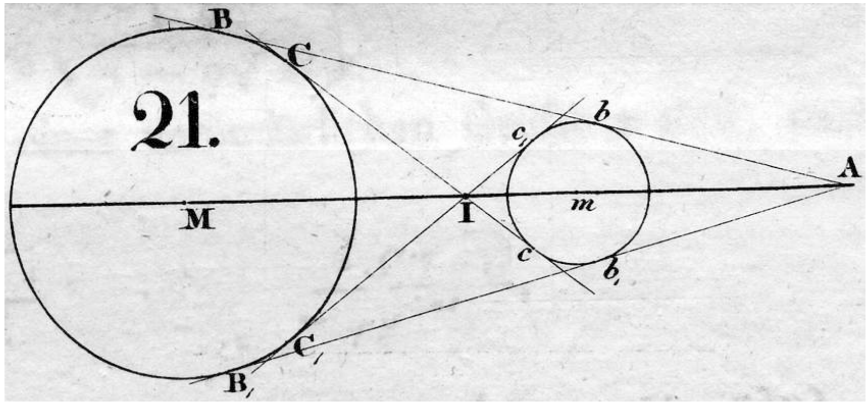
"Wir wollen diesen Punct $p(123)$ hinfort

'Punct der gleichen Potenz der drei Kreise M_1, M_2, M_3**'** nennen."

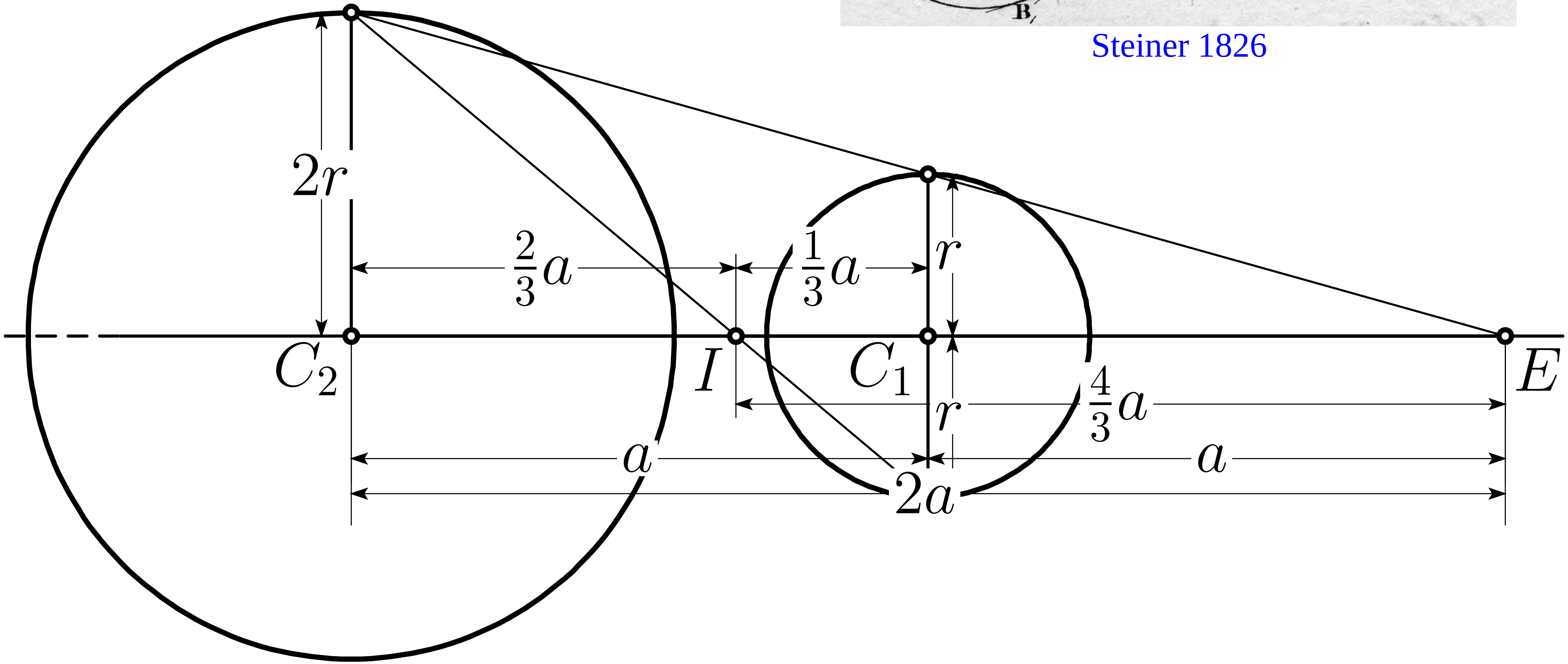
4. Similarity centers of two circles:



Fermat 1679

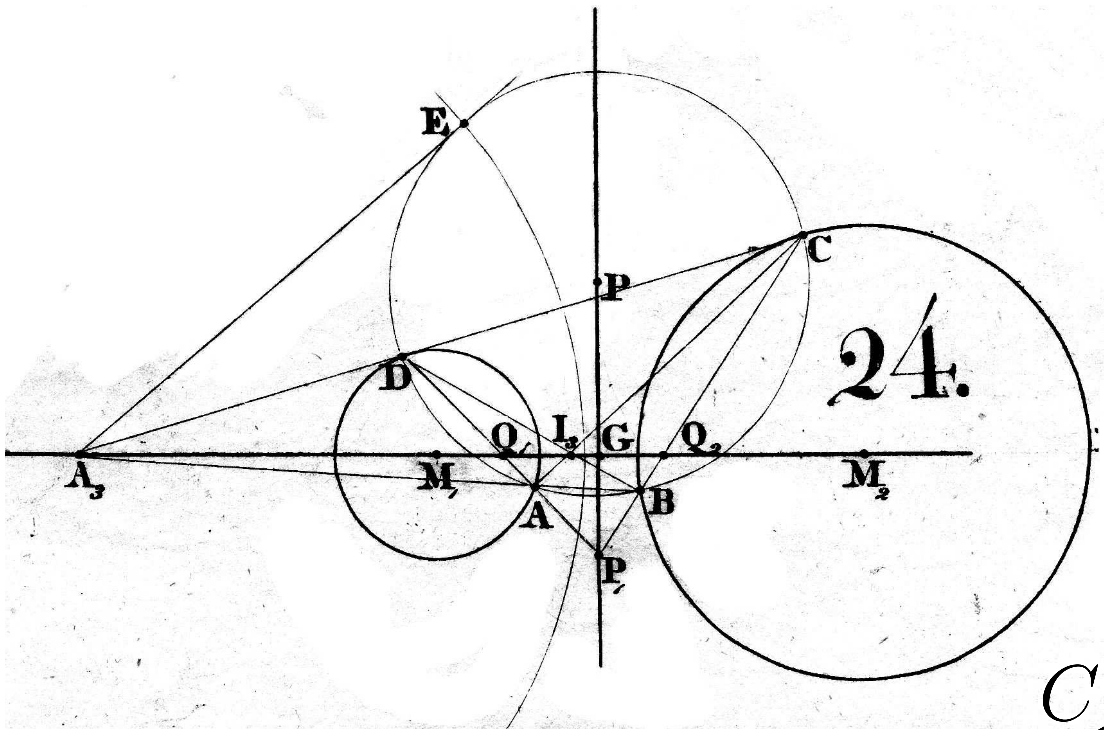


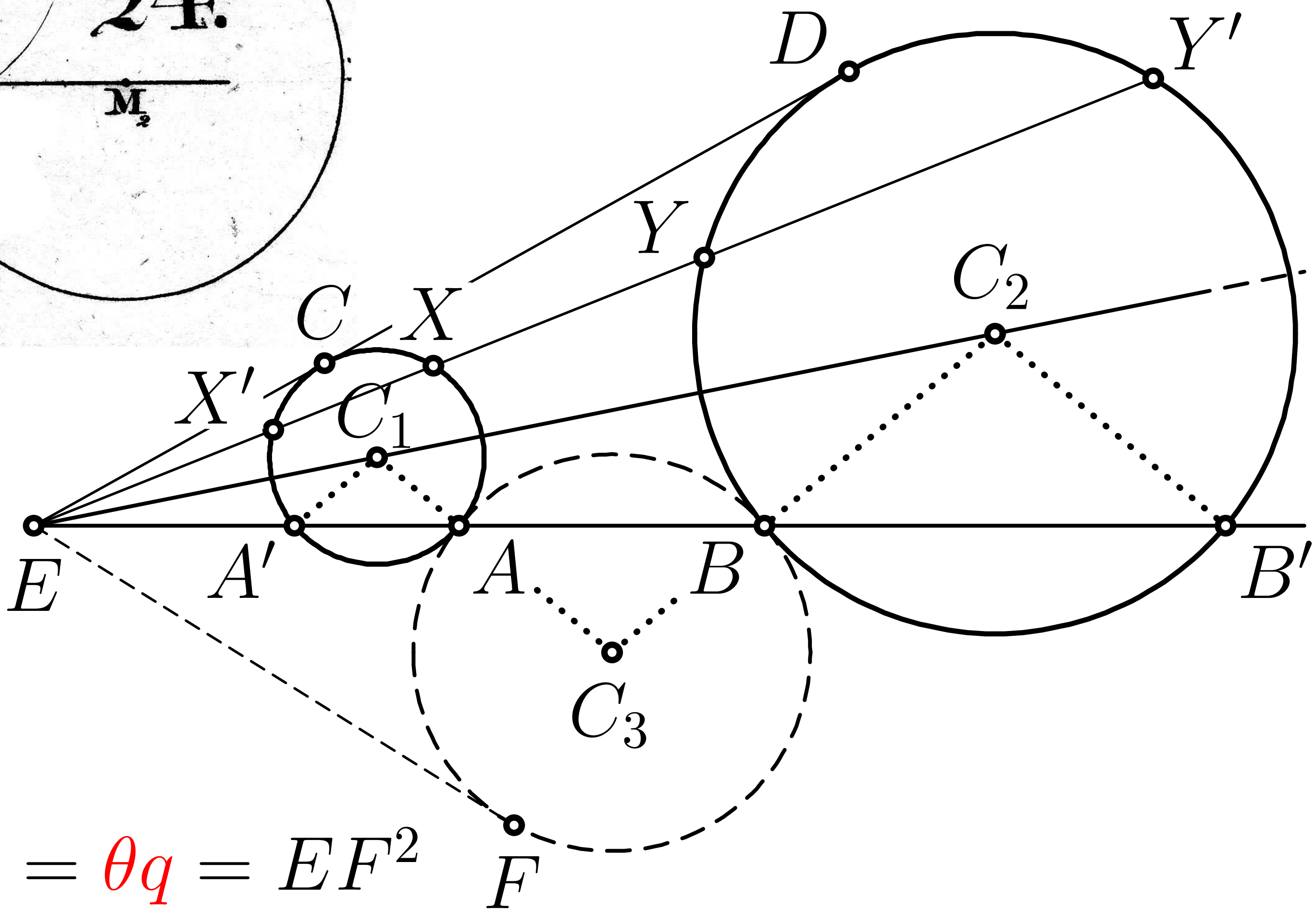
Steiner 1826



“Die Franzosen nennen diese Punkte *centre de similitude de deux cercles*. Wir haben diese Benennung zuerst aus einer Abhandlung von Euler genommen, siehe *L. Euler:* de centro similitudinis, 1777”.

5. Common power of two circles with respect to their similarity center.





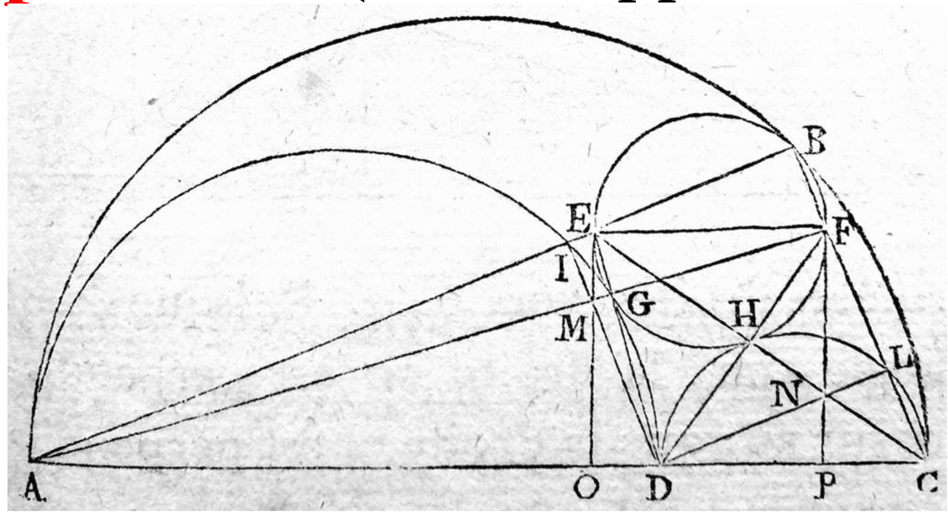
$$EC_2 = \theta \cdot EC_1$$

$$EX' \cdot EX = q$$

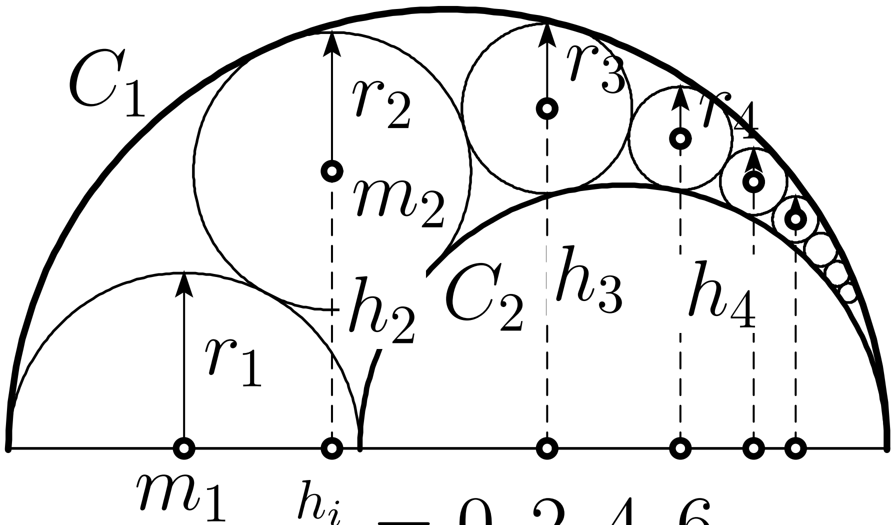
$$EX \cdot EY = EX' \cdot EY' = \theta q = EF^2$$

1. “Libris antiqua propositio” (“von Pappus überliefert”):

Archimedes’
“Lemma 6”

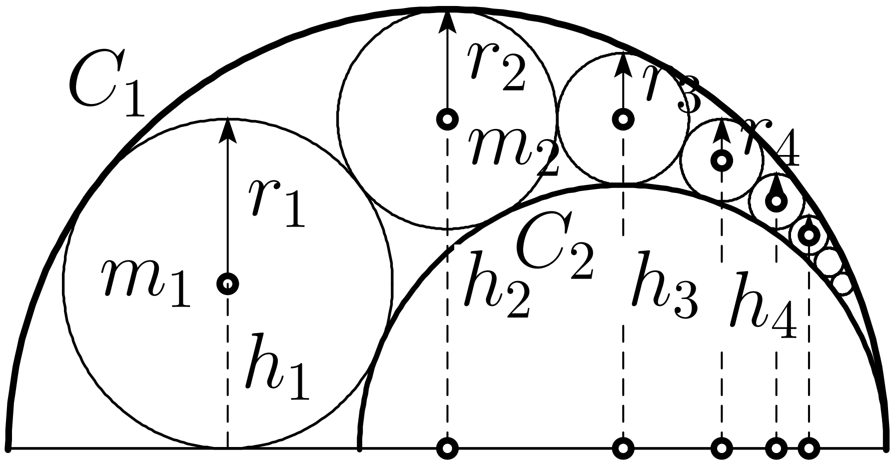


5 centuries later ...



$$\frac{h_i}{r_i} = 0, 2, 4, 6, \ldots$$

Pappus IV.16



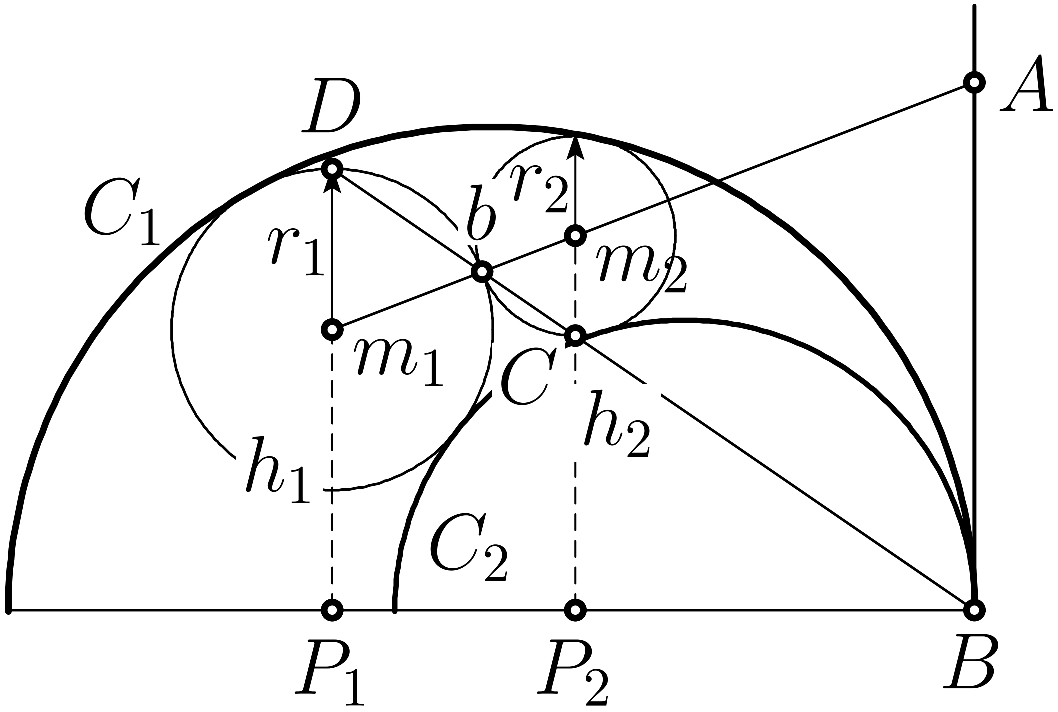
$$\frac{h_i}{r_i} = 1, 3, 5, 7, \ldots$$

Pappus IV.18

Steiner: “Den Pappischen Satz kannte ich nur ohne Beweis.”

Steiner's proof of Pappus' "antiqua" theorem:

(Crelle J. **1**, 1826, p. 261-262.)



(a) Tang. in B line of equal powers for C_1 and C_2, **2**;

(b) Sim. center A for m_1 and m_2 same power, $\Rightarrow$ on tang. **4,5**;

(c) Thales: $P_1B : P_2B = r_1 : r_2$;

(d) $AB^2 =$ com.pow. m_1, m_2 **5,1**;

(e) $Ab^2 = AX \cdot AY = AB^2$ **5**;

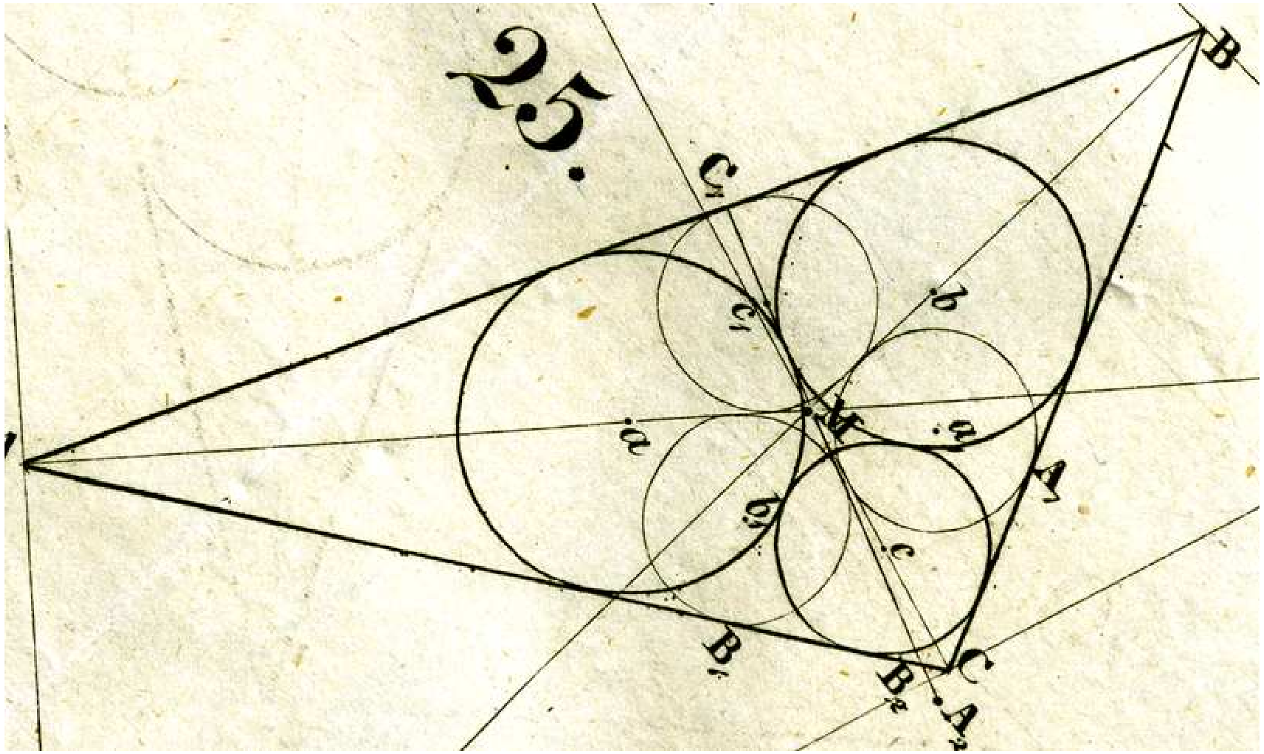
(f) $BAb \sim Cm_2b \sim Dm_1b$ $\Rightarrow \quad DbCB$ aligned;

(g) Thales and (c): $$\frac{h_1 + r_1}{r_1} = \frac{h_2 - r_2}{r_2} \quad \text{or} \quad \frac{h_2}{r_2} = \frac{h_1}{r_1} + 2\,. \qquad \square$$

Followed by many generalizations.

2. "Lösung der **Malfatti**schen Aufgabe..."

(Crelle J. **1**, 1826, p. 178.)



Problem:

Given triangle ABC

find 3 circles such that ...

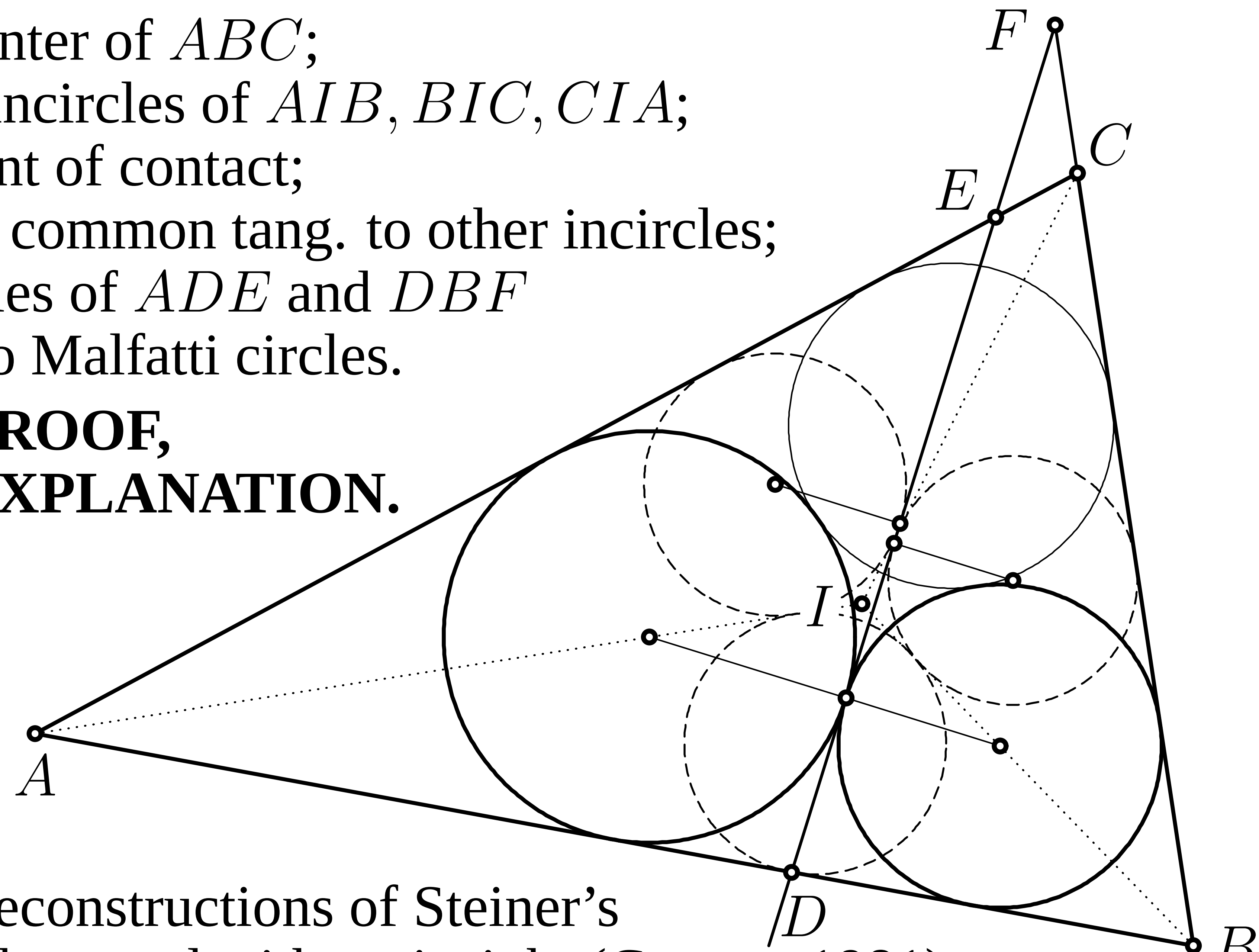
Problem appeared 1810/11 in Gergonne J;
"les rédacteurs des *Annales*" received letter from Italy that
"M. Malfatti, géomètre italien très-distingué"
had solved the problem, but "proof to long for this letter".

Proof was also to long for Malfatti's own publication
(Mem. Soc. It. d. sc., Modena 1803, **10**, p.235).

Steiner's geometric solution of the Malfatti problem:

I incenter of ABC;
draw incircles of AIB, BIC, CIA;
D point of contact;
DEF common tang. to other incircles;
incircles of ADE and DBF
$\Rightarrow$ two Malfatti circles.

NO PROOF,
NO EXPLANATION.



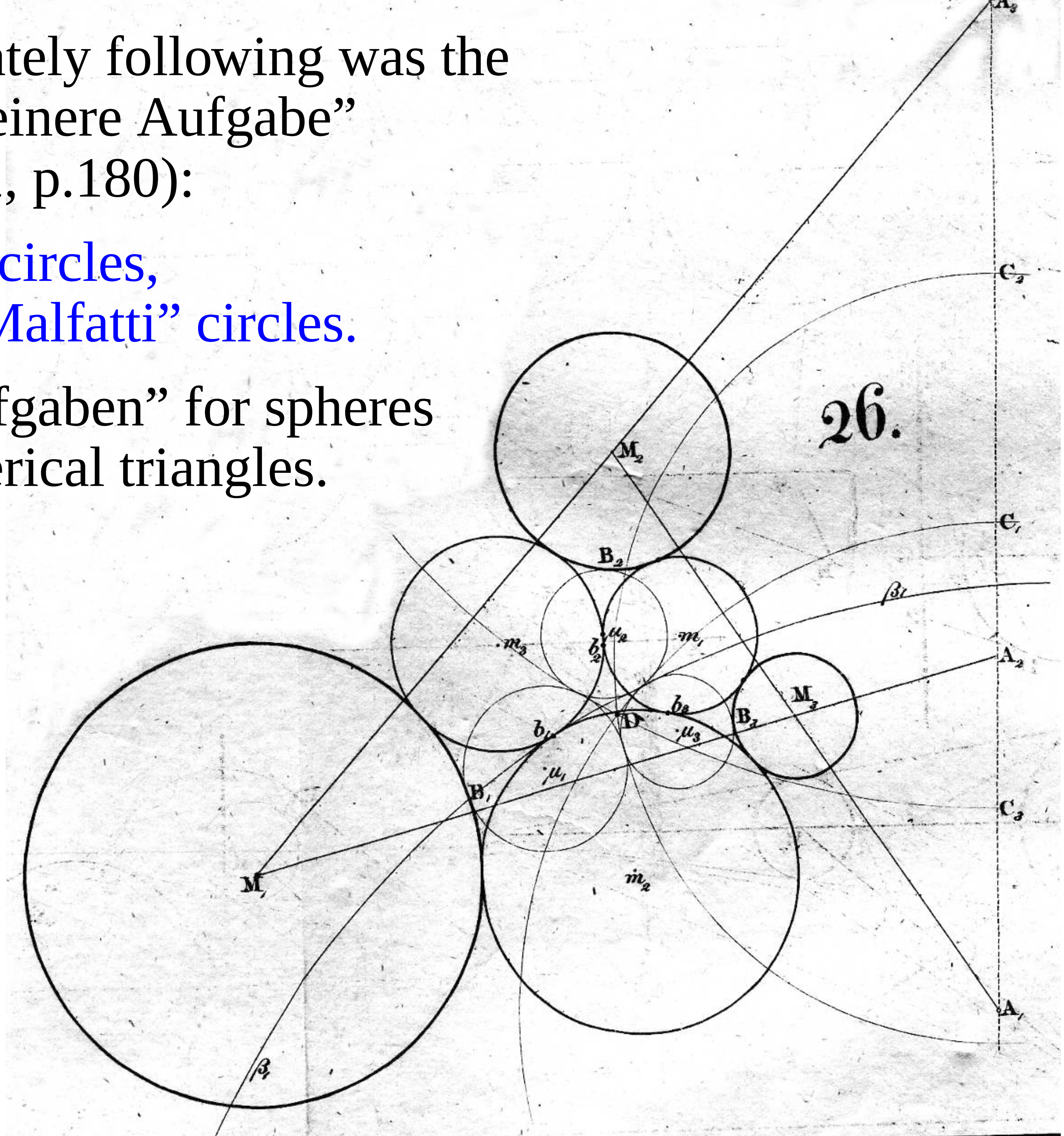
later reconstructions of Steiner's proof long and without insight (Carrega 1981).

Elegant trigonometric solution: K.Schellbach, Crelle 1853.

Immediately following was the
“allgemeinere Aufgabe”
(Crelle **1**, p.180):

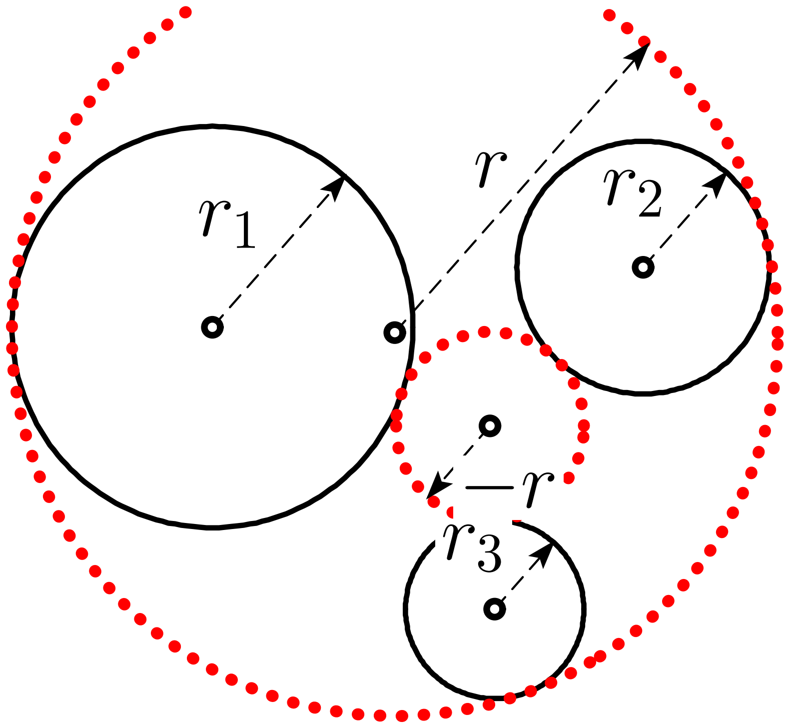
Given 3 circles,
find 3 “Malfatti” circles.

and “Aufgaben” for spheres
and spherical triangles.



3. Apollonius' three circle problem.

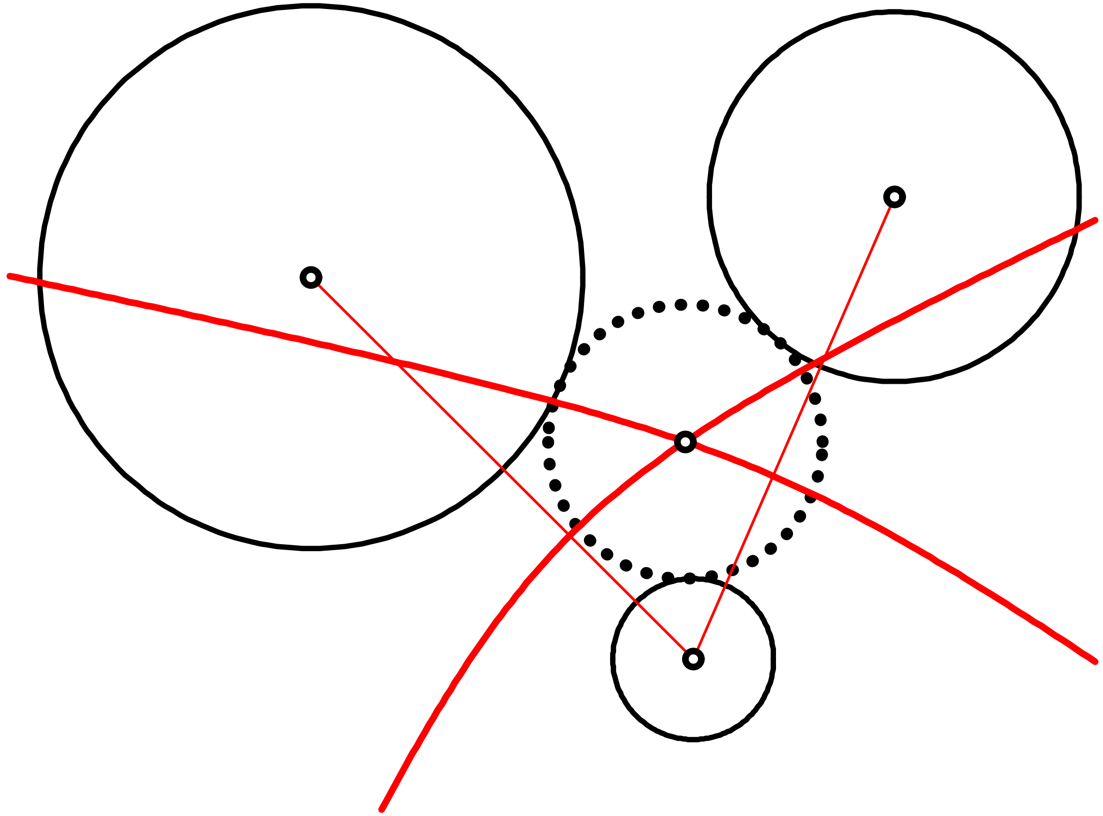
Viète (1595) challenged Adriaan van Roomen with the
Problem: Given 3 circles, find circles which touch all 3.



Pappus (Intro. to Book VII): reports on (lost) books of Apollonius *On Contacts* (11 probl., 21 lemmas, 60 thms !, "propositiones ... esse plures ... sed nos unam posuimus"):
Three objects given ("Punctis, et rectis lineis, et circulis") describe a circle tangent to all three ("datarum contingat").

Van Roomen answered: **easy !**

Find the center of the circle using 2 hyperbolas,
(the sets of points with same distance from 2 circles).

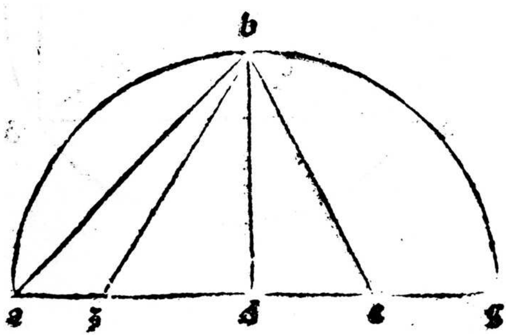


Viète: Remember the requirements of Greek culture ...

Three Types of Constructions (Pappus, p. 7):

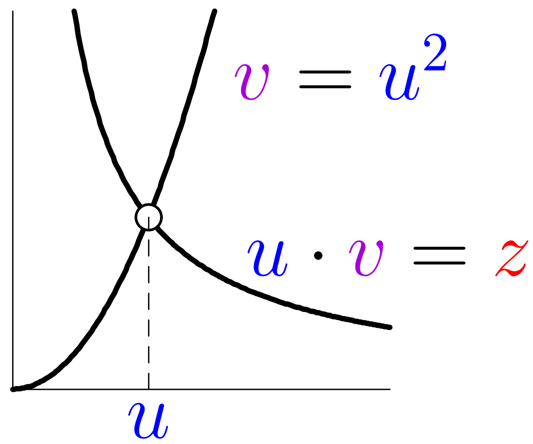
Problematum geometricorum antiqui tria genera eſſe ſtatuerunt, & eorum alia quidem plana appellari, alia ſolida, alia linearia. Quę igitur per rectas lineas, & circuli circumferentiam ſolui poſſunt, merito plana dicuntur; etenim lineæ, per quas eiuſmodi problemata ſoluuntur, in plano ortum habent. Problemata vero quæcunque ſoluuntur, aſſumpta in conſtructionem aliqua coni ſectione, vel pluribus; ſolida appellantur namque ad conſtructionem neceſſe eſt ſolidarum figurarum ſuperficiebus, nimirum conicis vti. Reſtat tertium genus, quod lineare appellatur: Lineæ enim aliæ præter iam dictas in conſtructionem aſſumuntur, varium, & tranſmutabilem ortum habentes, quales ſunt helices, & quas græci τετραγωνίζουσας appellant, nos quadrantes dicere poſſumus, conchoides, & ciſſoides, quibus quidem multa, & admirabilia accidunt. Cum igitur tales ſint problematum differentiæ, antiqui geometræ problema an

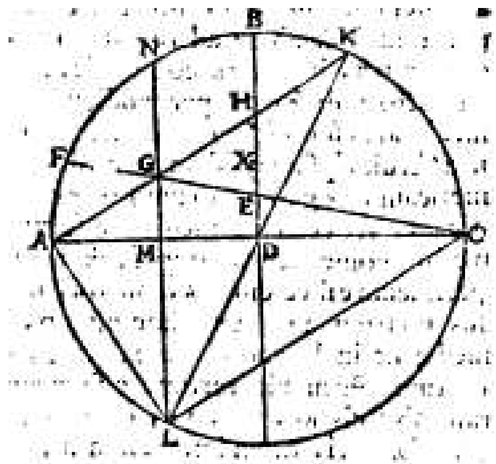
- "plane" problems ($\varepsilon\pi\acute{\iota}\pi\varepsilon\delta\alpha$): with ruler and compass;
- "solid" problems ($\sigma\tau\varepsilon\rho\varepsilon\acute{\alpha}$): using conics;
- "grammical" problems ($\gamma\rho\alpha\mu\mu\iota\kappa\acute{\alpha}$): using other curves.



b.ʒ.equale lateri penthagoni.

(Ptolemy, Almagest, ed. 1496)





François Viète (1600): “clarissime Apolloni Belga”

FRANCISCI VIETÆ

APOLLONIVS GALLVS.

Seu,

EXSUSCITATA APOLLONII PERGÆI

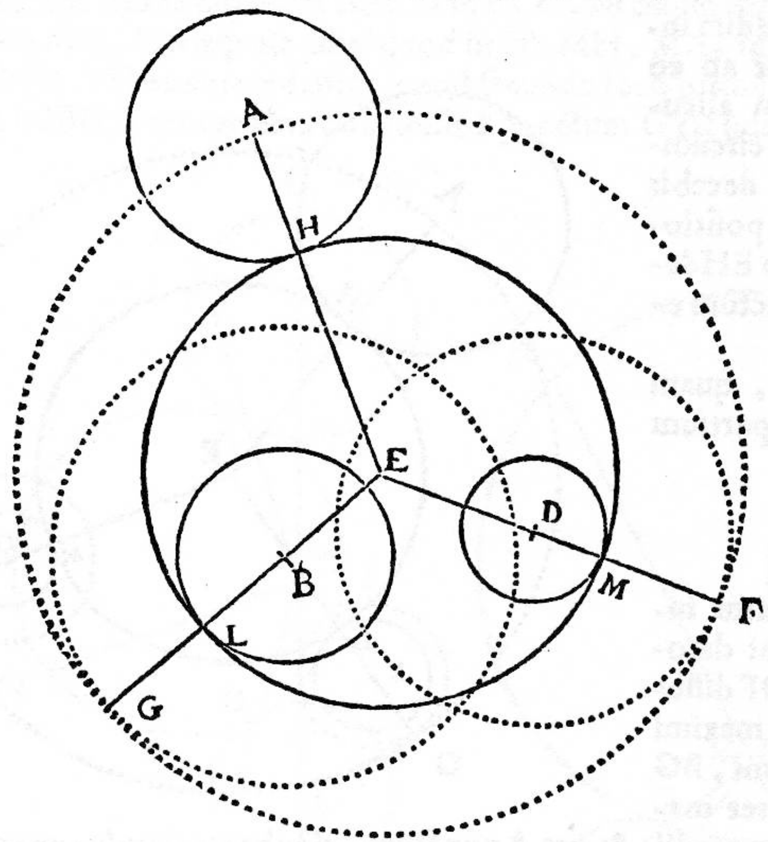
ΠΕΡῚ ΕΠΑΦΩ͂Ν GEOMETRIA.

Ad V. C.

ADRIANUM ROMANUM Belgam.

PROBLEMA Apollonii de deſcribendo circulo, quem tres dati contingant (clariſſime Adriane) Geometrica ratione conſtruendum propoſui φιλομαθέσι, non Mechanice. Dum itaque circulum per hyperbolas tangis, rem

(Viète, *Apollonius Gallus* 1600, Opera p. 325)



Vt qua
mo tanga
& à ſecun
tio extra.
adgregat
diametro
mi & tert
gregatum
metroru
ſecundi.
punctum
tur circu
poſititii
tingant.

Vt qua
mo tanga
& à ſecun

(Viète, *Apoll. Gallus* 1600, Opera p. 338)

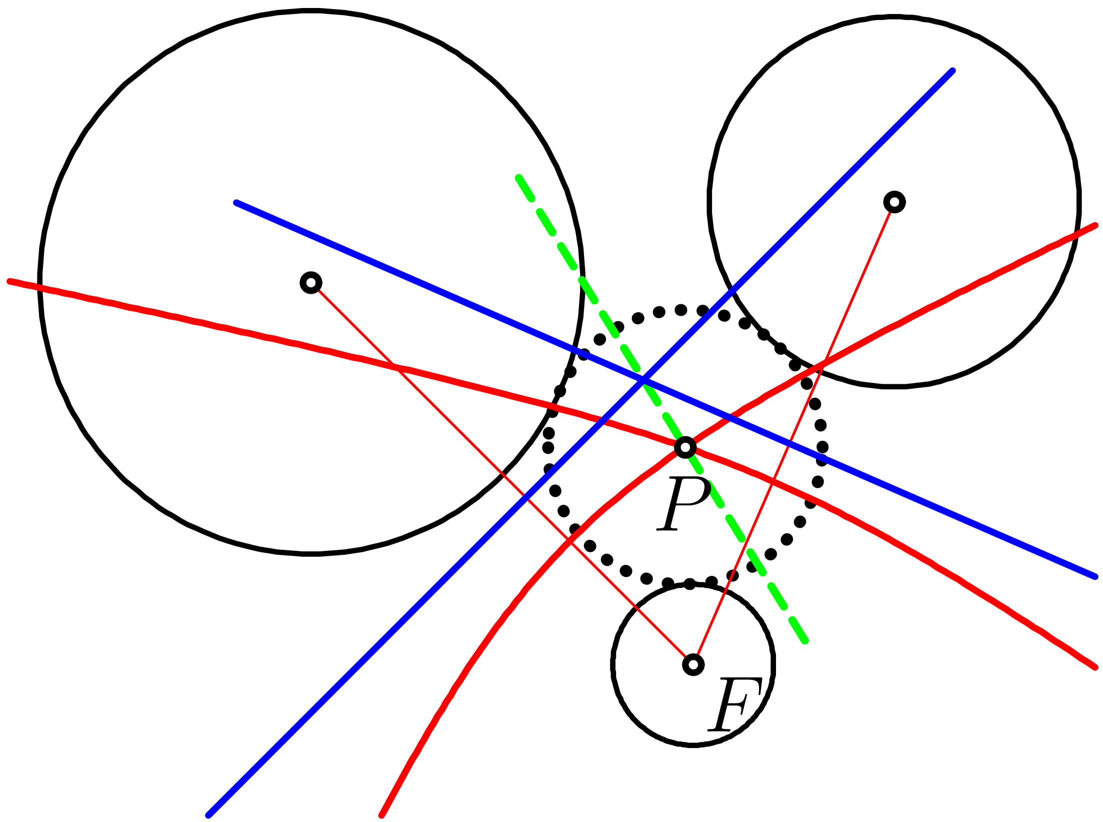
Reclamaret Euclides, & tota Euclideorum ſchola. Ergo clariſſime Adriane, ac ſi placet Apolloni Belga, quoniam Problema quod propoſui planum eſt, tu vero ceu ſolidum explicaſti, neque ideo occurſum hyperbola-

(Viète, *Apollonius Gallus* 1600, Opera p. 325)

the problem is planar, no hyperbolas required !!

Viète solves Pappus’ 10 problems geometr. one after the other.

Surprise: Newton. (*Principia*, ed.1726, Lemma XVI).



van Roomen's solution is planar too !!

- because both hyperbolas have a common focus;
- by using Pappus VII.238: $PF = \dfrac{1}{e_i} \cdot \text{dist. } P\text{-directrix}_i$; gives additional linear equation.

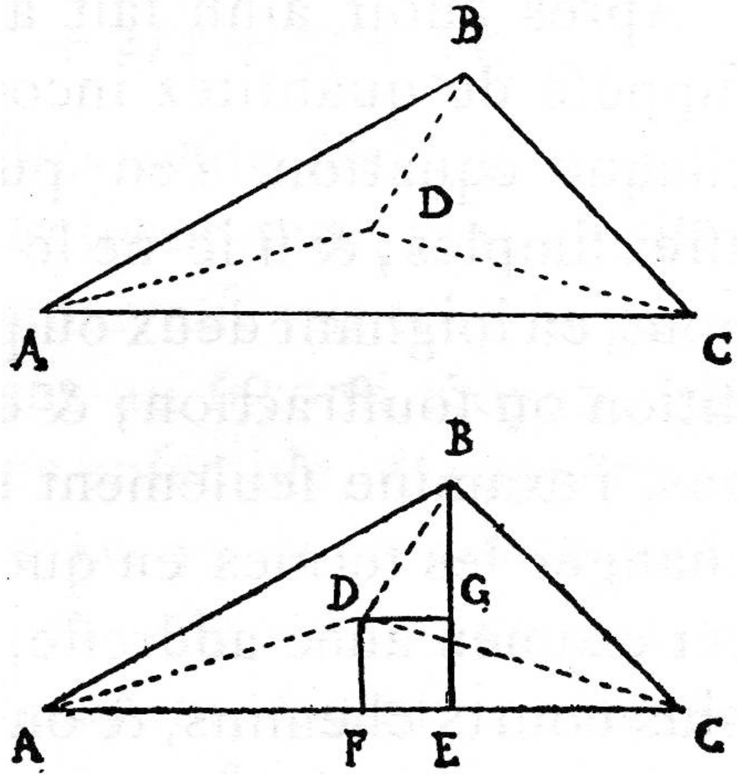
Descartes. (1643): lettres to Elisabeth, princess of Bohemia.

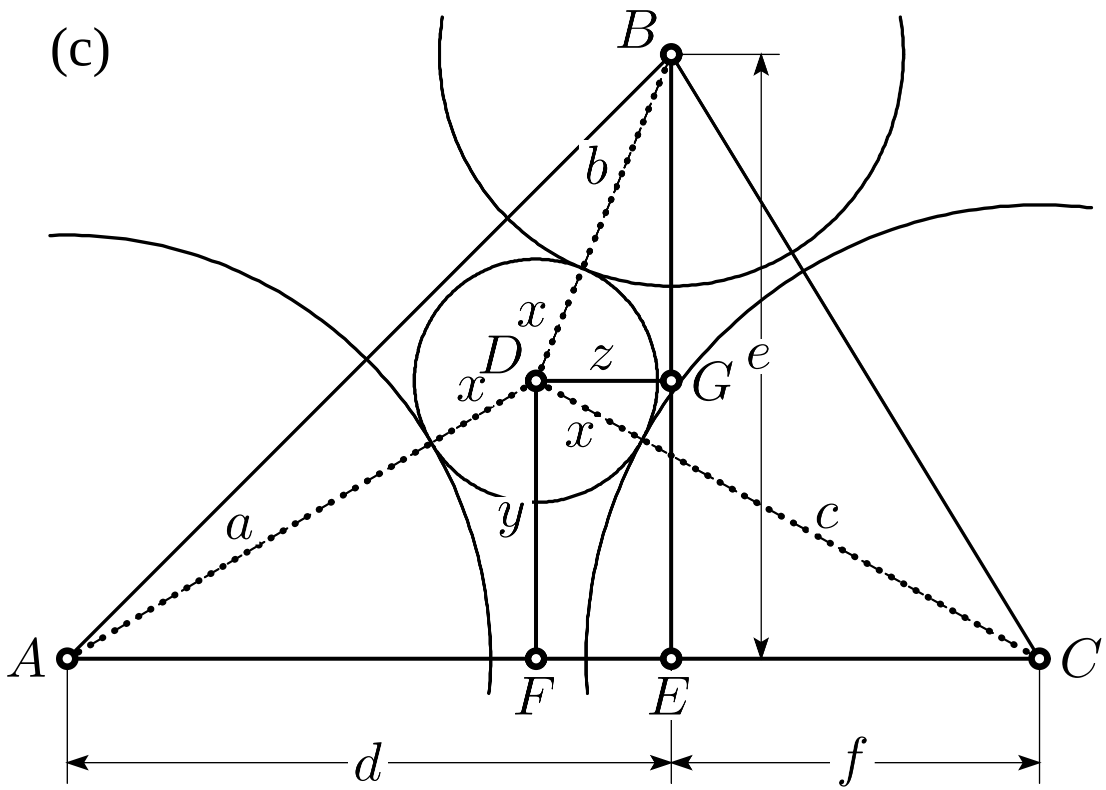
Mathematical problems are best remedy for cultivating the human mind.

Three circle problem of Apollonius is best math. problem.!

“Vostre Altesse” did not answer for a while ...

Desc.: Problem difficult; even an Angel finds solution only by miracle!





Desc: slant lines in above pict. ... terrible calc. ... no result before three months;

Desc: intr. orthog. lines below. unknowns x, y, z. ... **(birth orth. Cart. coord!)**

Pythagoras: 3 quadratic eqns; subtract ... linear equs;‘-> **“le Probleme est plan”**.

all these tedious calculations, “trop ennuyeux à Vostre Altesse” (-> Gergonne)

Gergonne's solution

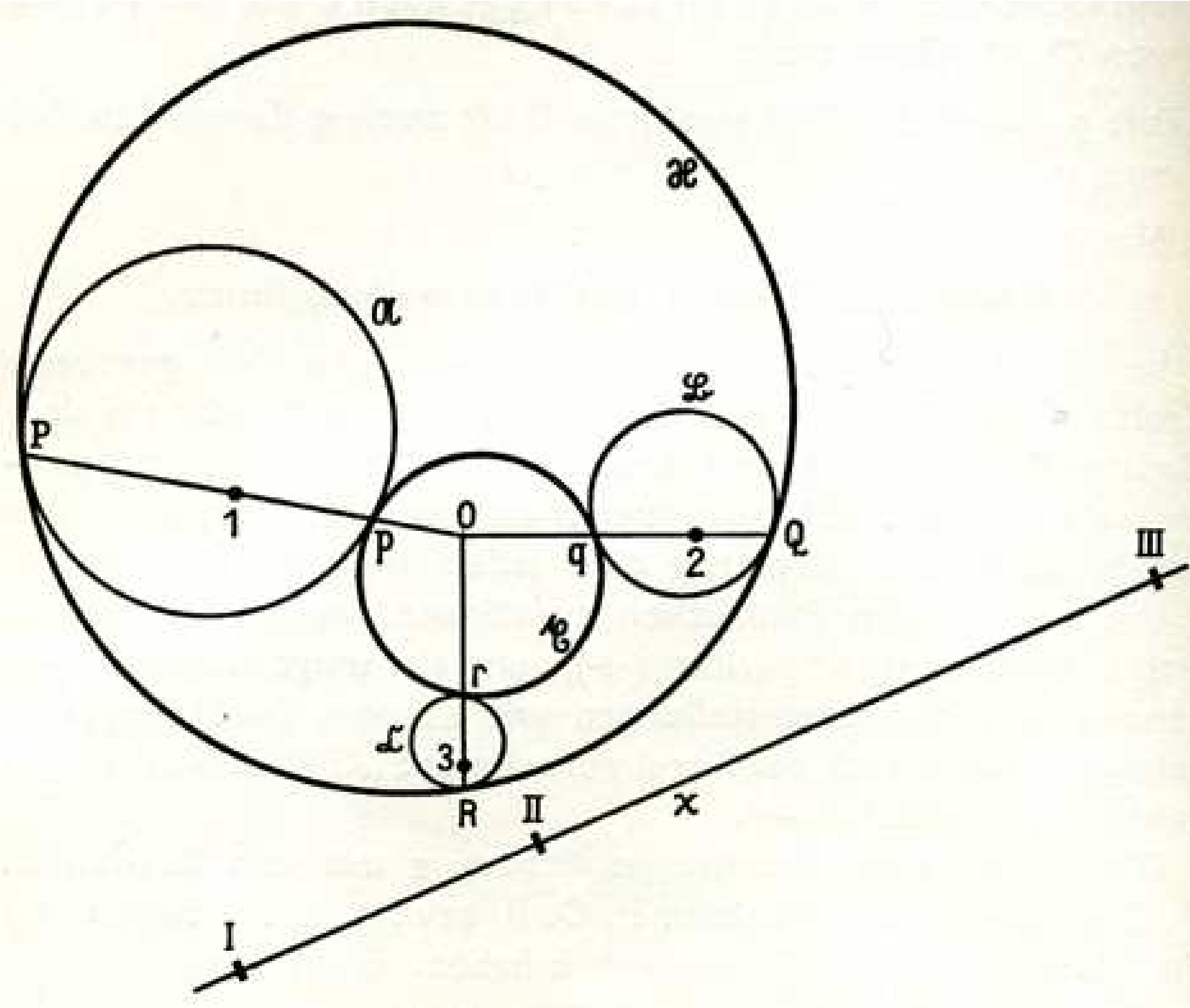
(1813; fig. from Dörrie 1932)

(I,II,III $=$
ext. sim. centers)

($1, 2, 3 =$
poles of similarity axis)

($O =$ point of
common power.)

Proof by tedious
analytic calculations
(see Descartes).



Jakob Steiner: (> 300 pages) *Allgemeine Theorie über das Berühren und Schneiden der Kreise und Kugeln*, from the years 1825-1826, unpublished unti 1931, > 300 pages; discusses Apollonius 3-circle problem in detail (p.175):

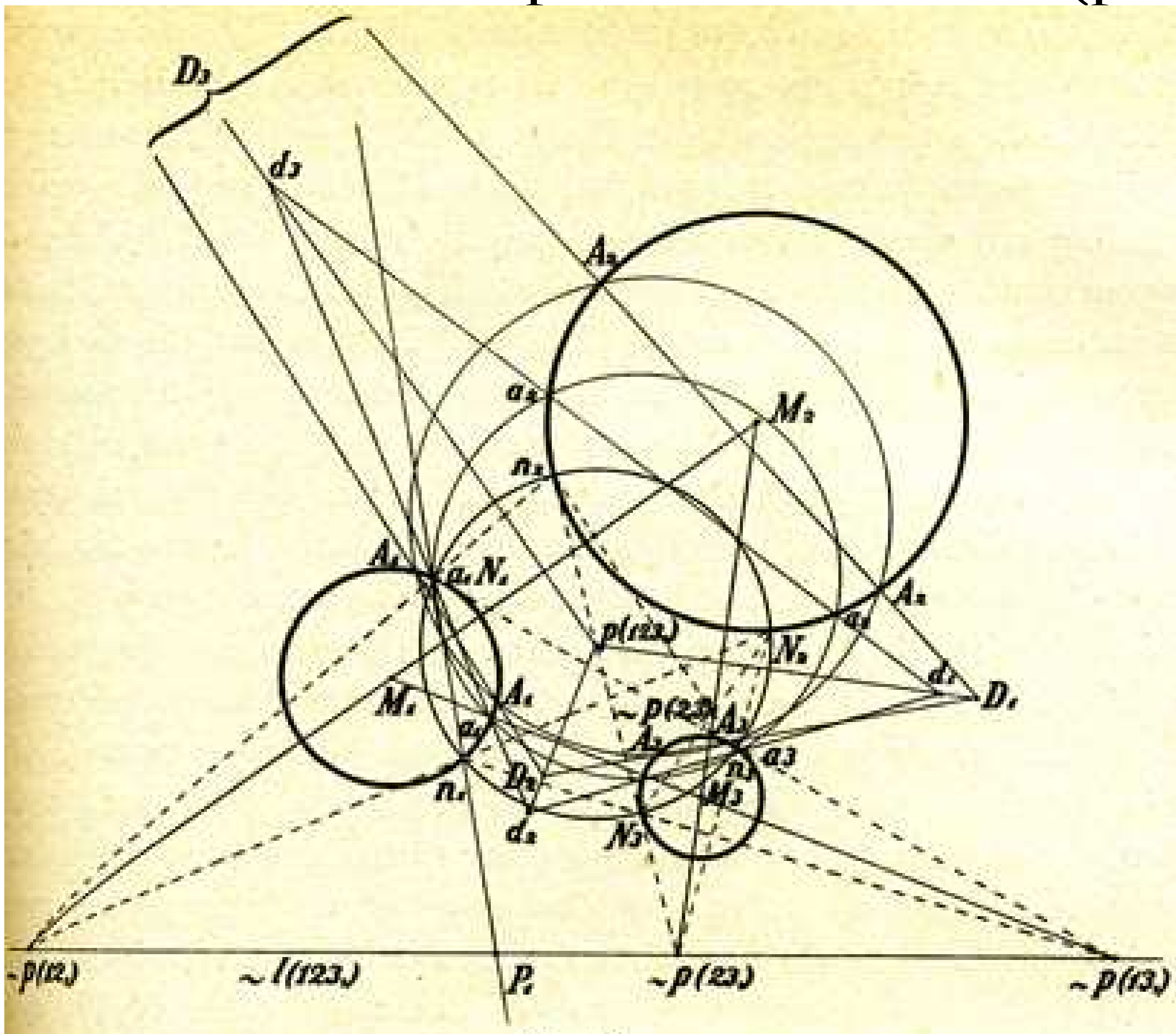
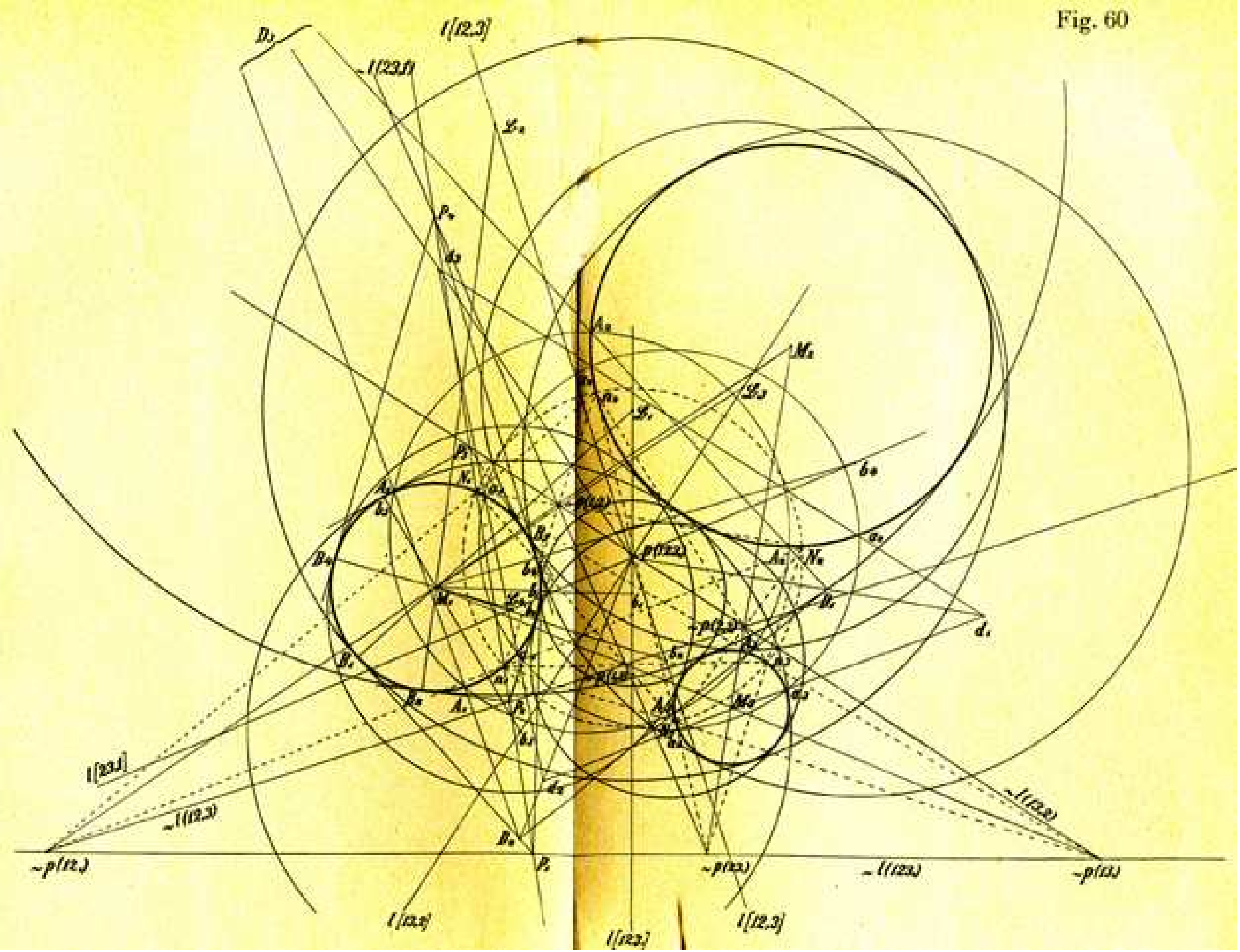


Fig. 48

"... ein Muster von Eleganz und Einfachheit" (R.Fueter, F.Gonseth 1930) complete algorithm (drawn by Dr. F. Züllig, 1930)



Similar to Gergonne, but elegant geom. proof and **all 8** sols.

1936, Frederick Soddy, Nobel in chemistry, publ. in *Nature*:

The Kiss Precise

FOR pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

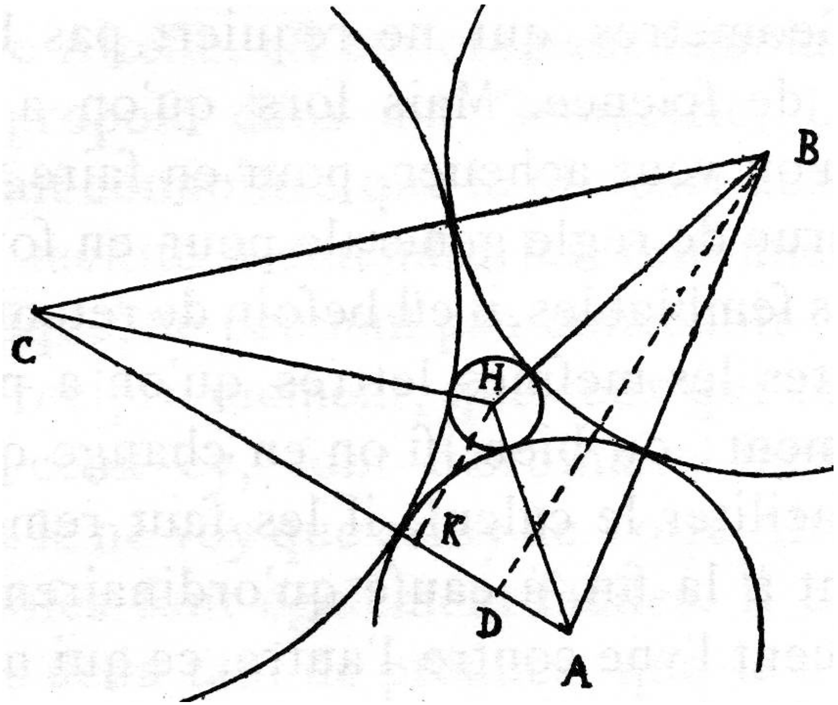
To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. Soddy.

Understand ? Go back again three centuries...

Descartes. (Nov. 1643): next letter to Elisabeth.

Desc: try the simpler case where three given circles of radius a, b, c touch each other.



Descartes' drawing, 1901 edition of Œuvres, vol. 4, p. 48, BGE Ca1227/4

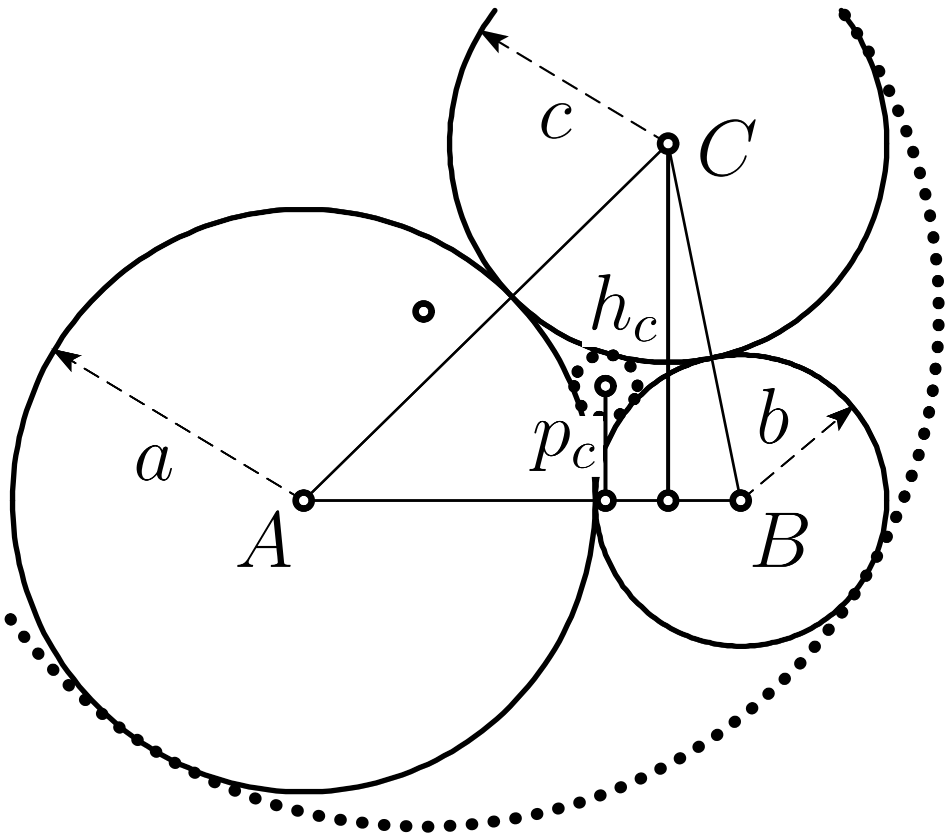
Here, says Descartes, "Vostre Altesse" would arrive at this equation

$$a^2b^2c^2 + a^2b^2x^2 + a^2c^2x^2 + b^2c^2x^2 =$$
$$2\left(abc^2x^2 + ab^2cx^2 + a^2bcx^2 + ab^2c^2x + a^2bc^2x + a^2b^2cx\right)$$

(without proof !) or, as we write it today

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{x^2} = 2\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{1}{ax} + \frac{1}{bx} + \frac{1}{cx}\right) .$$

Steiner's proof. (Crelle **1**, p. 273):



Pappus' "ancient" thm: $\dfrac{p_c}{x} = \dfrac{h_c}{c} + 2$

or $\dfrac{p_c}{h_c} = x\left(\dfrac{1}{c} + \dfrac{2}{h_c}\right)$;

"Known formula" $\dfrac{p_a}{h_a} + \dfrac{p_b}{h_b} + \dfrac{p_c}{h_c} = 1$

(Euler) gives

$$1 = x\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{2}{h_a} + \frac{2}{h_b} + \frac{2}{h_c}\right);$$

Insert Eucl. I.41: $\dfrac{2}{h_c} = \dfrac{a+b}{\mathcal{A}}$ and $\sim$ for a, b;

Finally insert Heron's formula for $\mathcal{A}$:

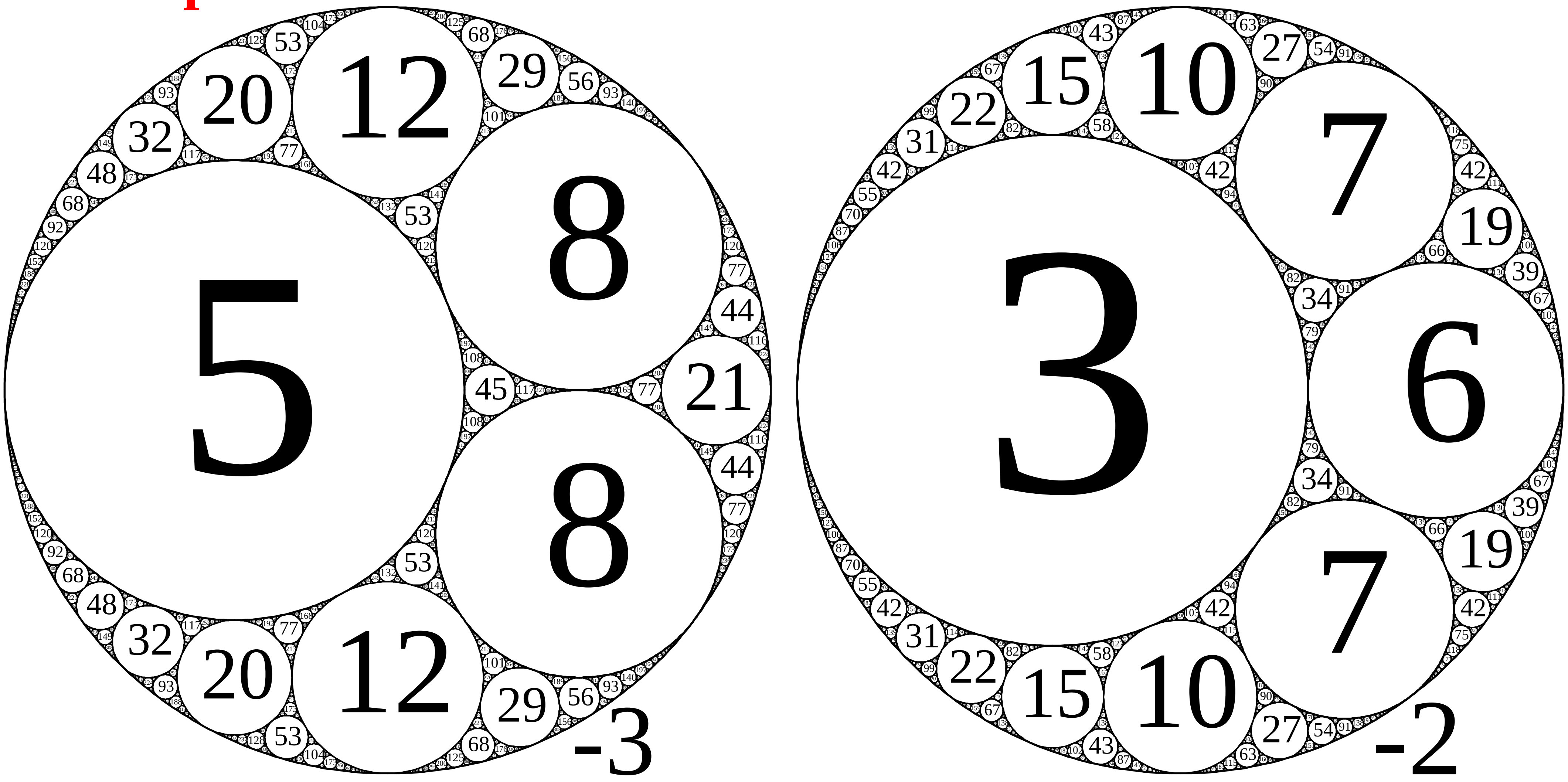
$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{b+c}{\mathcal{A}} + \frac{c+a}{\mathcal{A}} + \frac{a+b}{\mathcal{A}}$$

$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{2(a+b+c)}{\sqrt{(a+b+c)abc}} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \pm 2\sqrt{\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}}.$$

Formula becomes simpler if radii are replaced by inverses (the "bends" of Soddy).

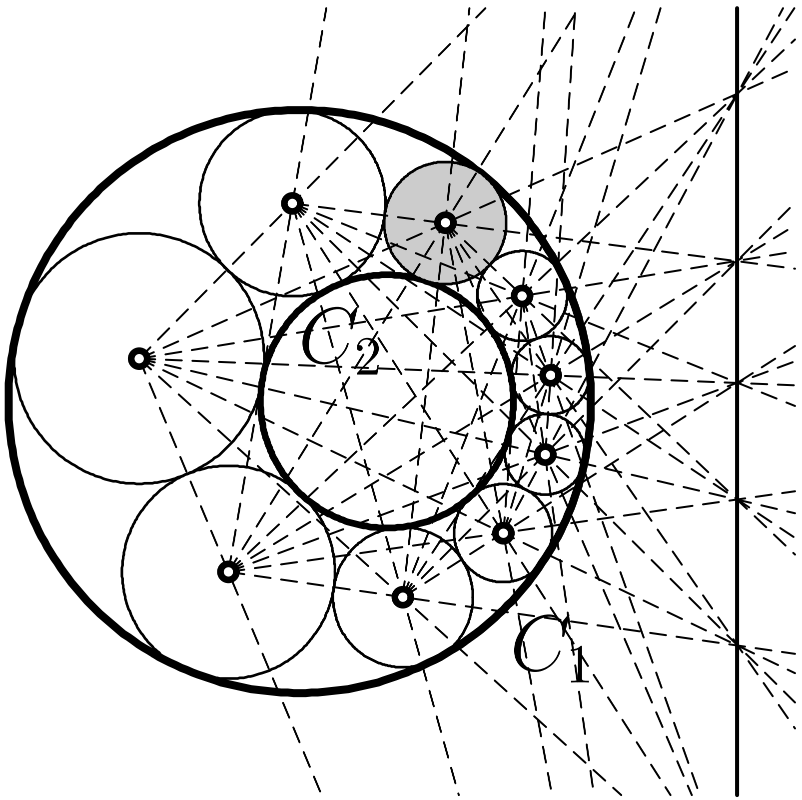
“Apollonian packings” (Lagarias, Mallows and Wilks 2002): Nice discovery: if four consecutive “bends” are integers, then the $\sqrt{\ldots}$ is integer (use formula forwards and backwards), then **all** bends remain integer.

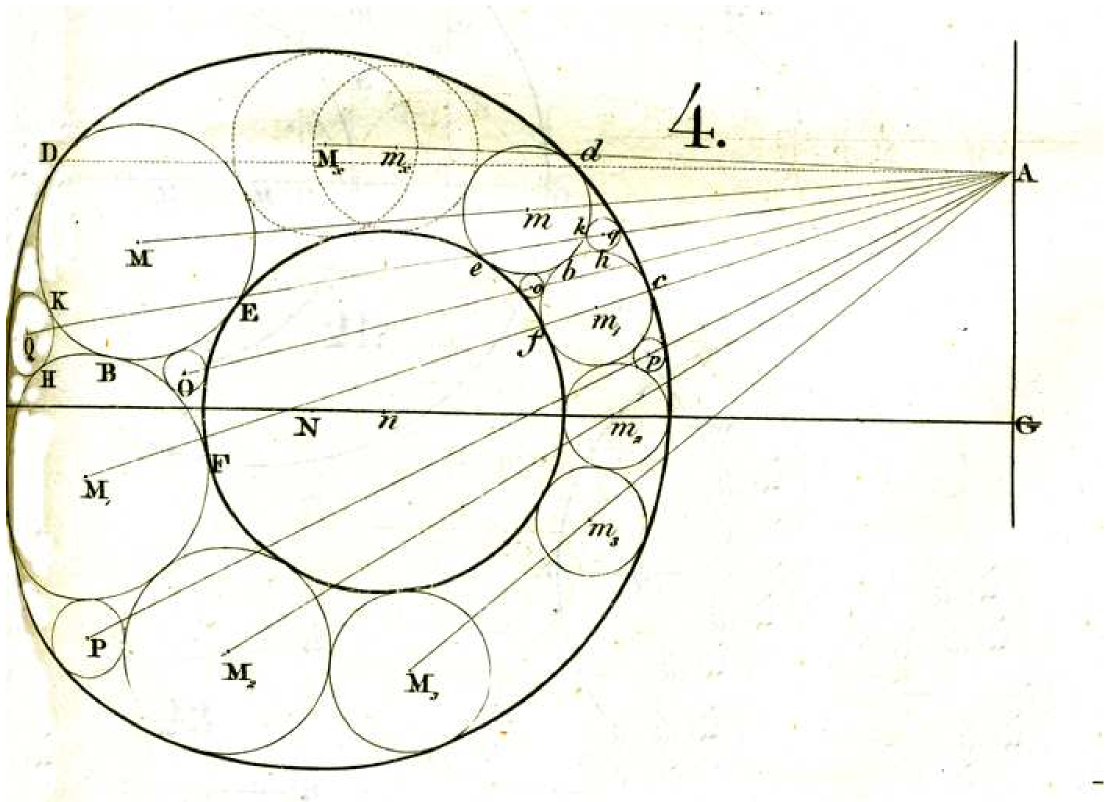
Examples.



Towards Steiner's Porism.

What happens with the "Shoemaker's knife" if the two circles are no longer tangent ? Then the line of common power is no longer tangent:



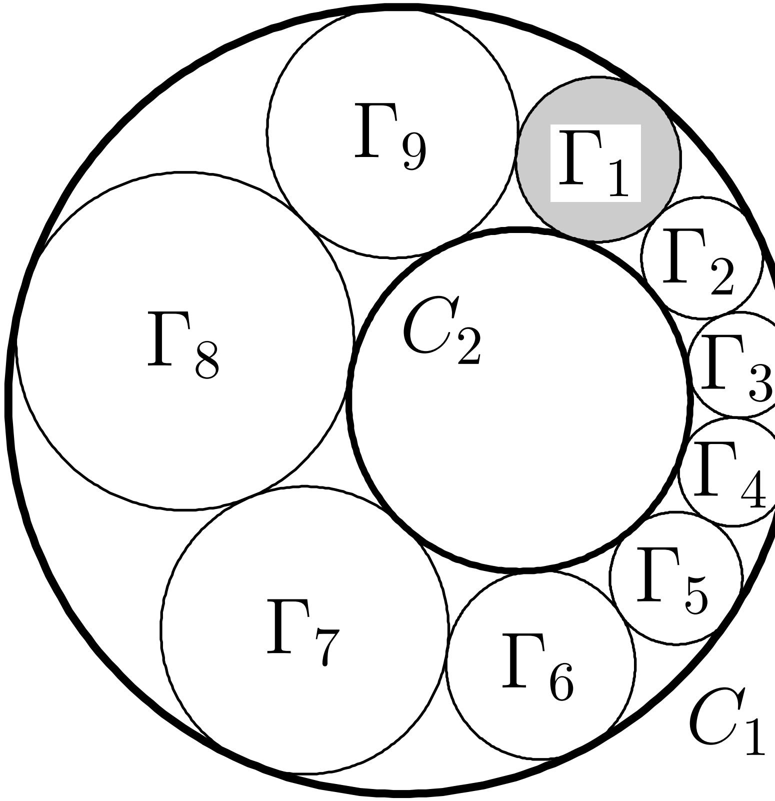


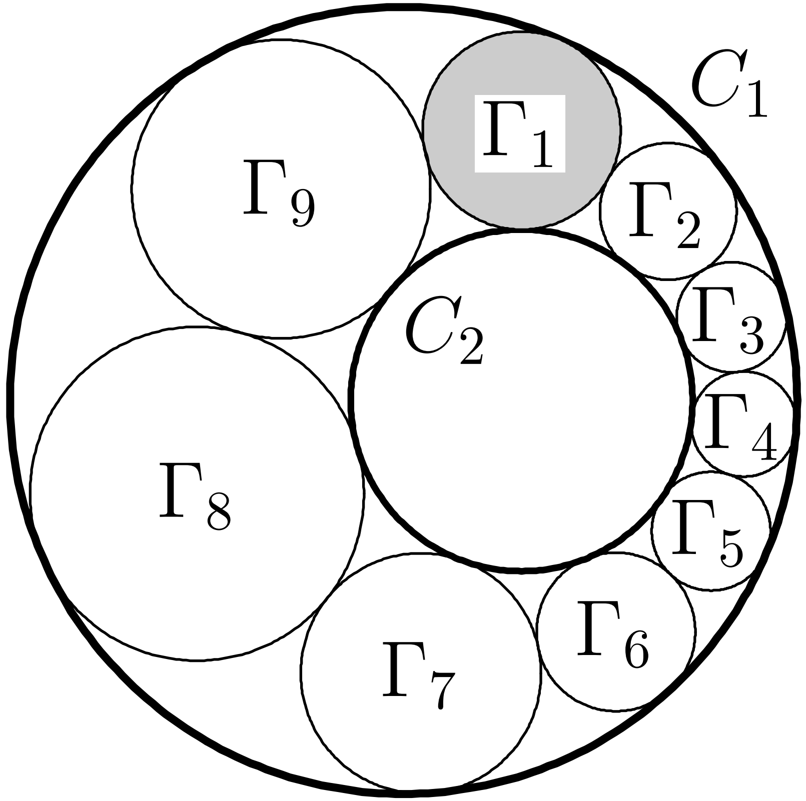
Steiner 1826, Crelle Journal **1**, p.288

Long search for an interesting theorem also in this case ...

Steiner's Porism (Crelle **1**, 1826, p.254, Art.22)

"... nachstehende merkwürdige Folgerungen ziehen."

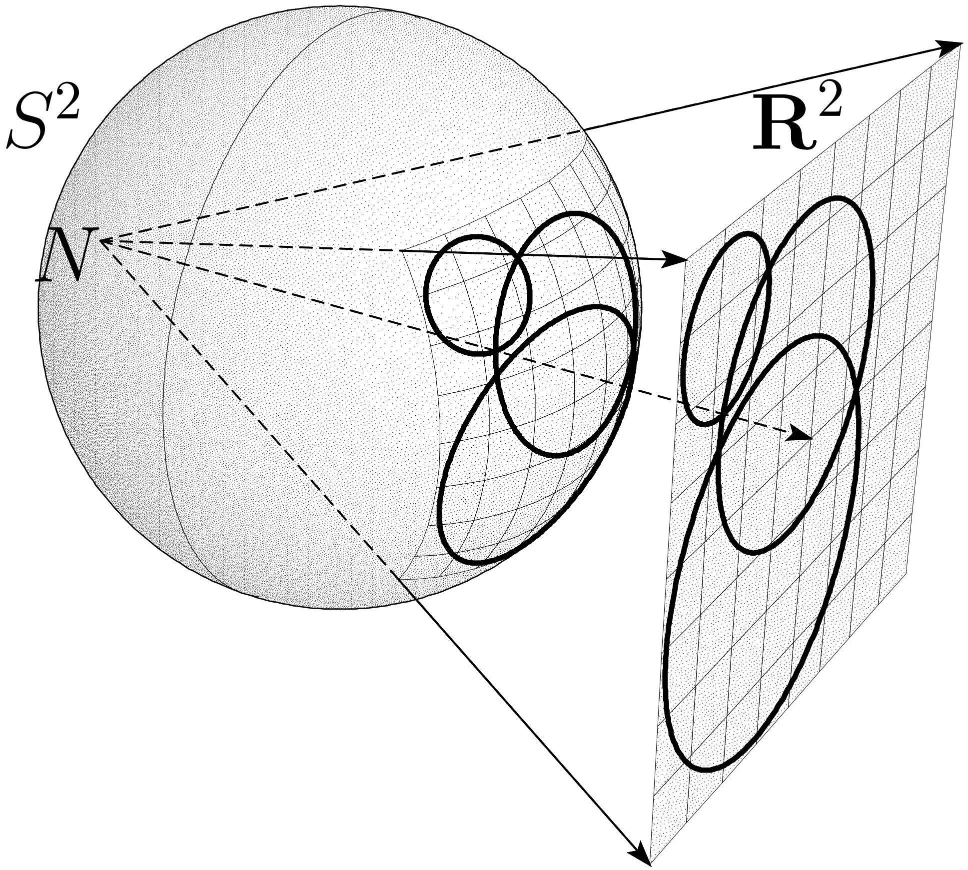


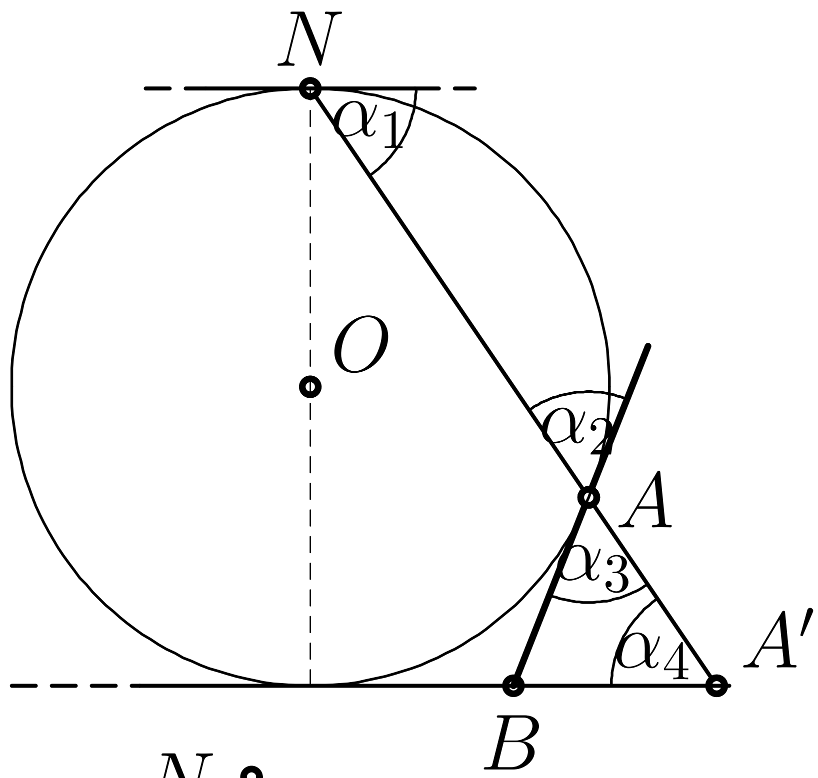


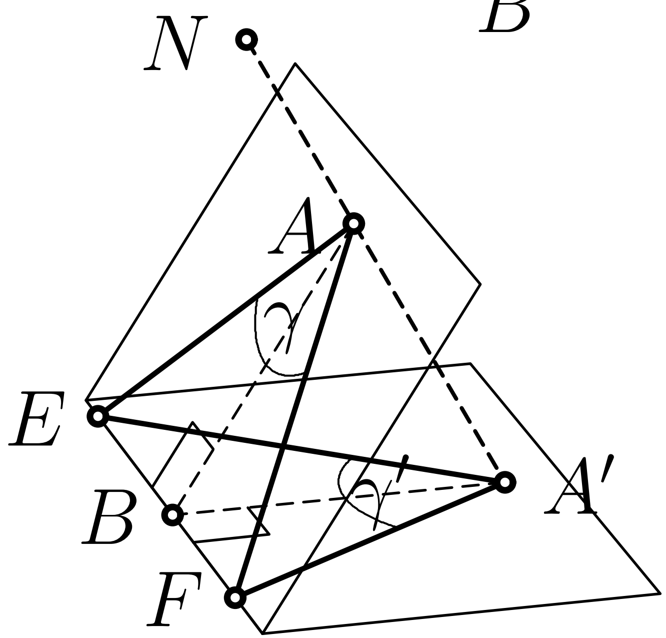
If the space between two fixed circles C_1 and C_2 is filled with circles $\Gamma_1, \Gamma_2, \ldots$ all touching each other and the circles, and if for some n $\Gamma_{n+1} = \Gamma_1$, then the same holds for **any** initial position of Γ_1.

Exists simple proof with inversion map; known to Steiner ???

Excursion: Stereographic projection.

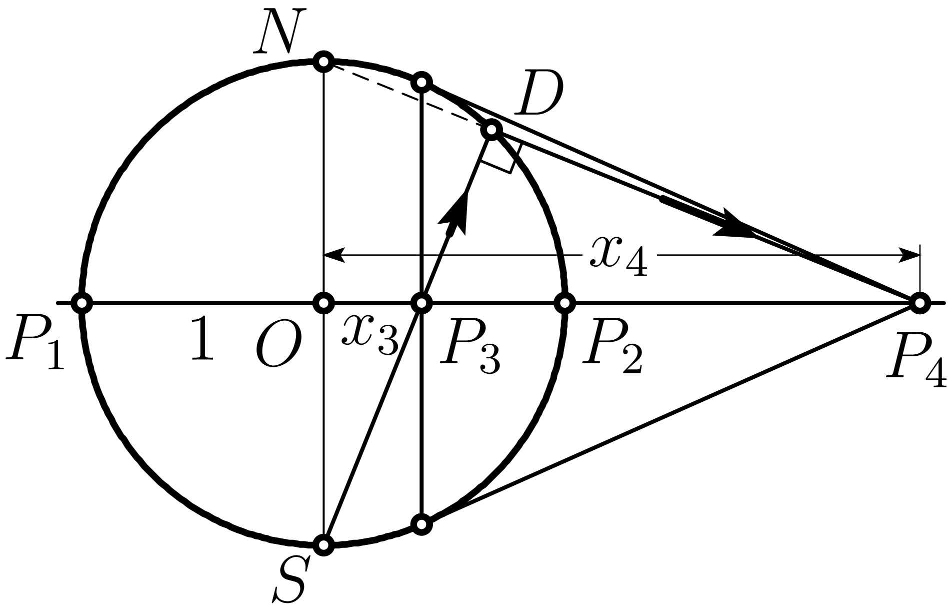






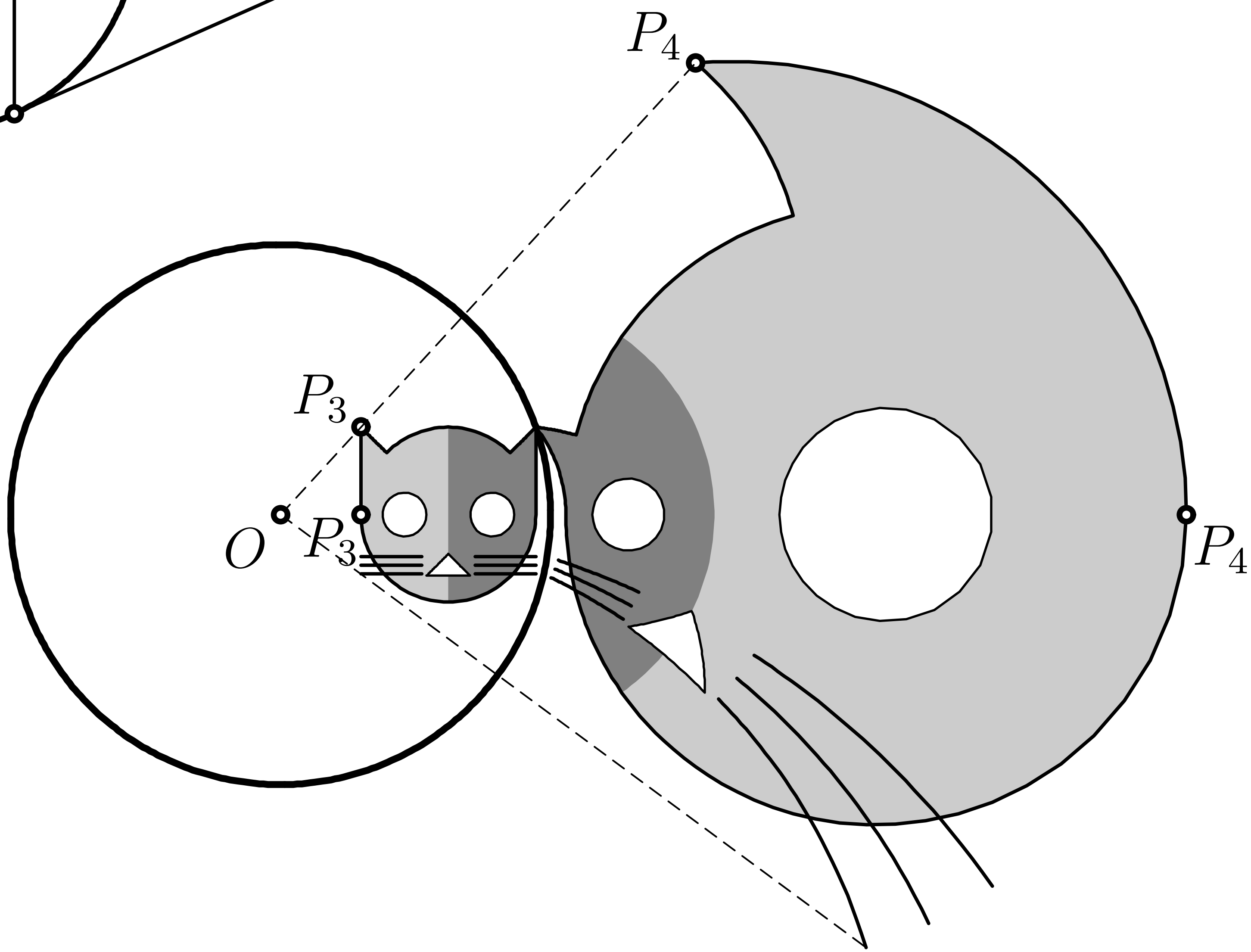
Proj. from pole to equator plane:

- maps circles to circles (Ptolemy)
  (rediscovered Miquel 1838)
- preserves angles (Halley 1696)



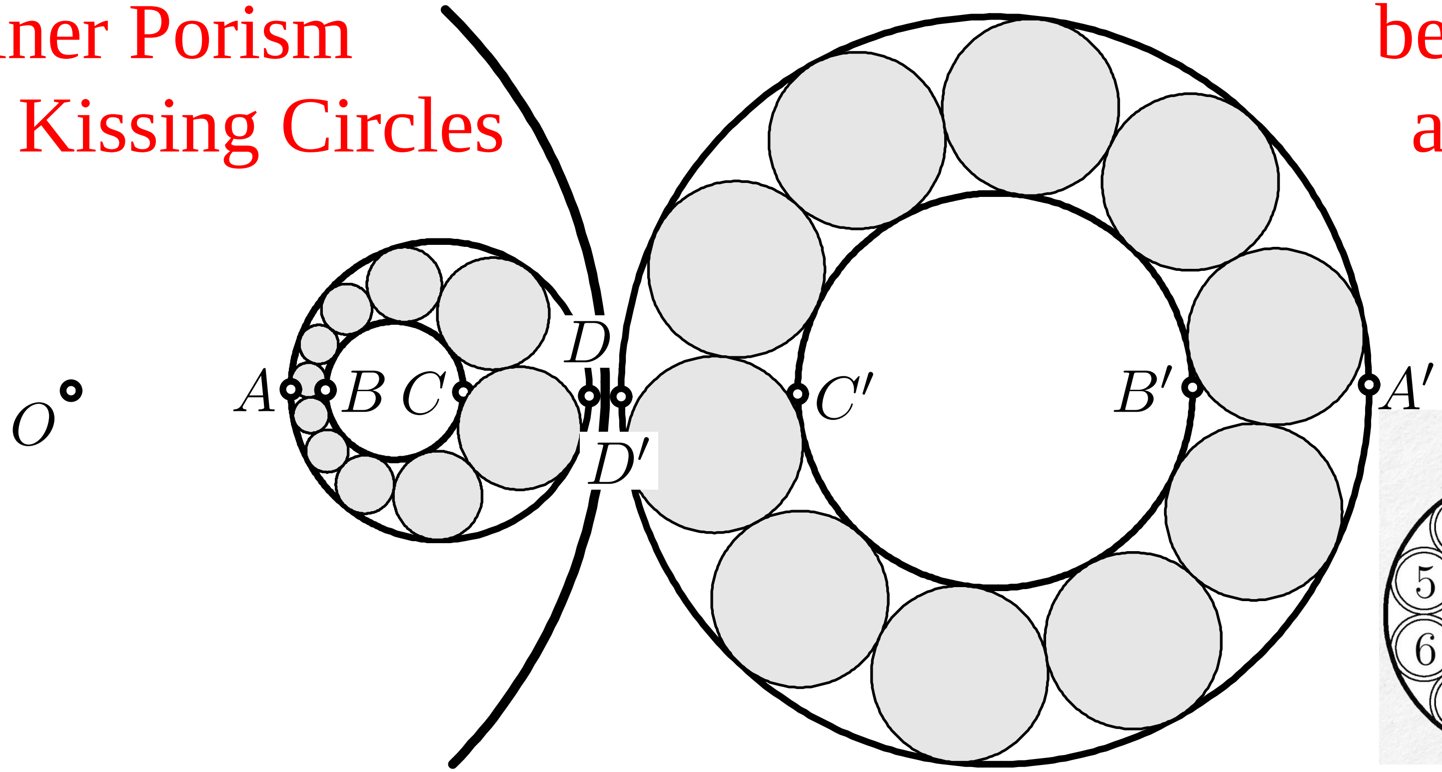
Inversion Map $P_3 \mapsto P_4$:

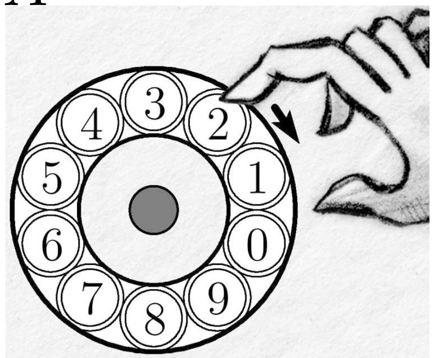
composition stereogr. map $S \to P_3 \to D$
with stereogr. map $N \to D \to P_4$
$\Rightarrow$ preserves circles and angles !
$\Rightarrow OP_3 \cdot OP_4 = R^2, \quad P_1P_2P_3P_4$ harm.

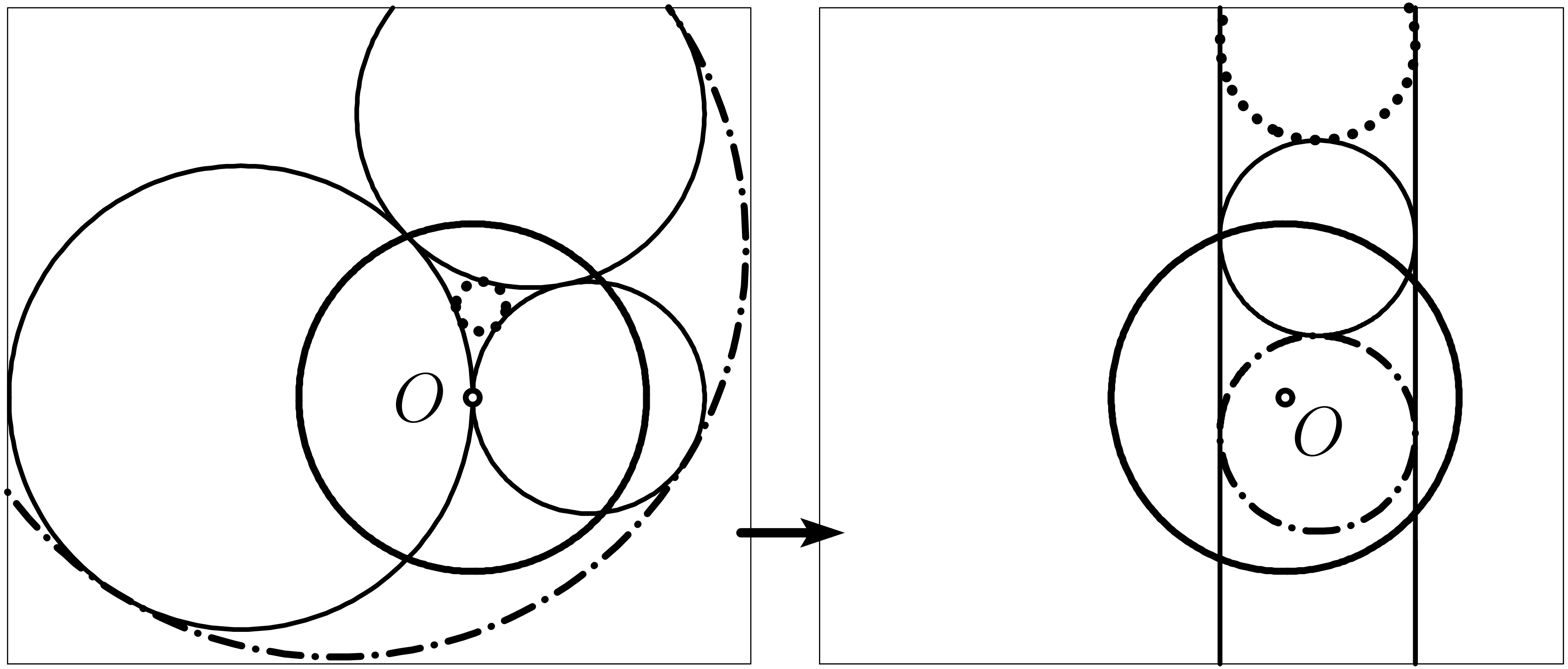


Steiner Porism and Kissing Circles

become triviality after inversion !

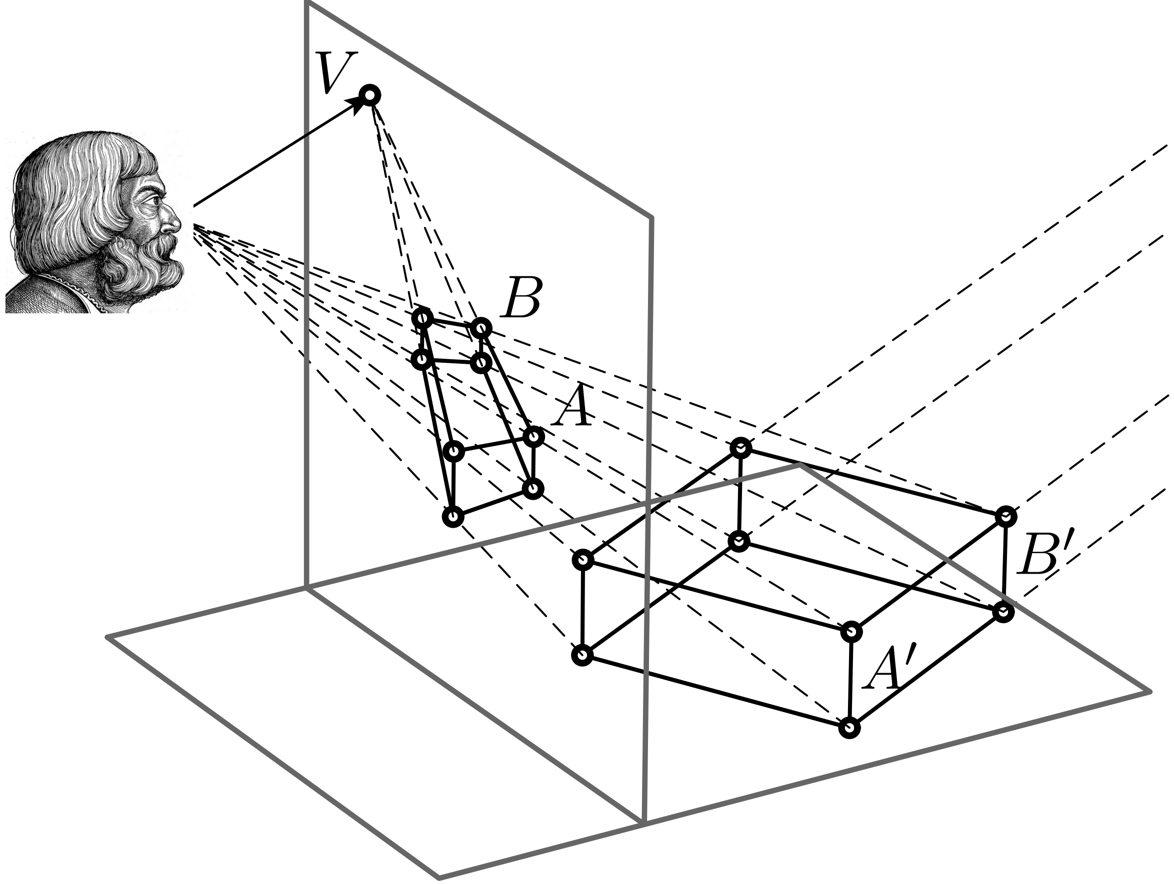




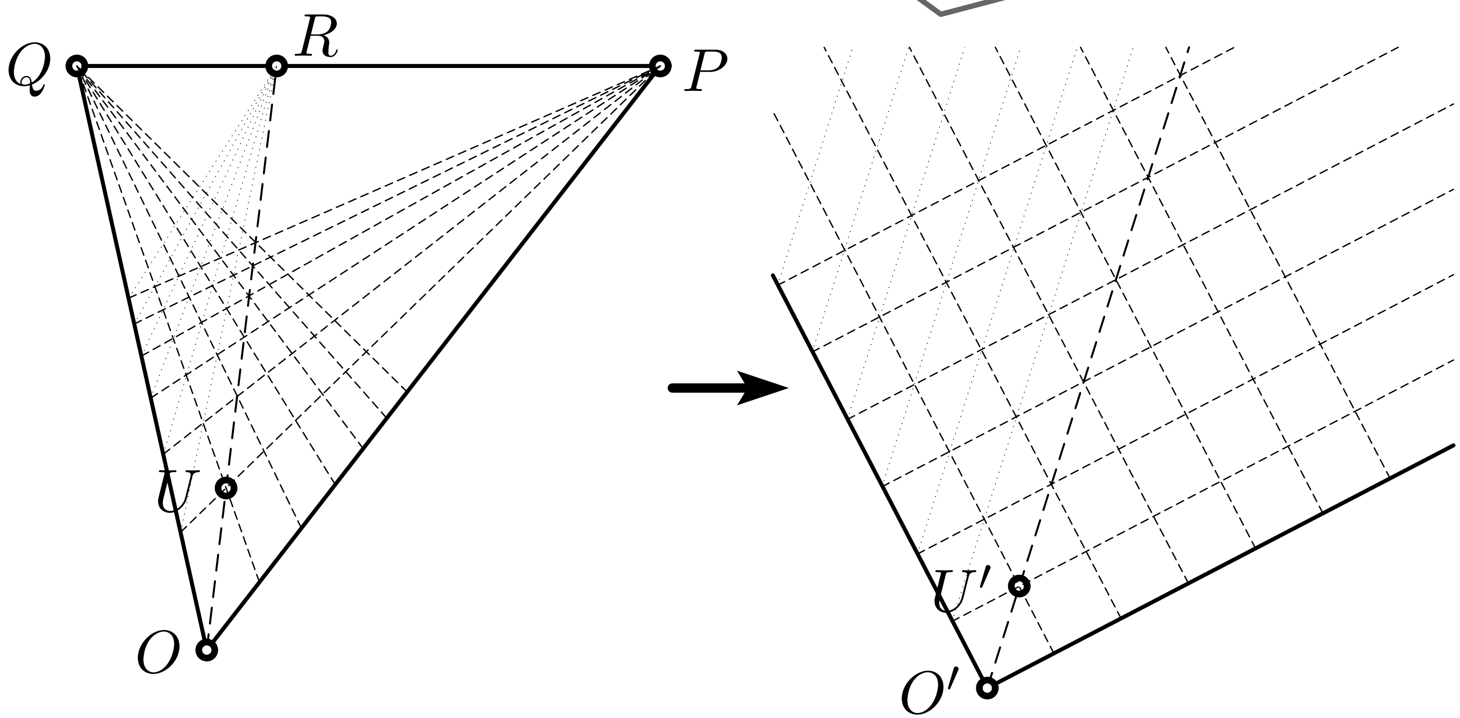


Another Excursion:
J.-V. Poncelet

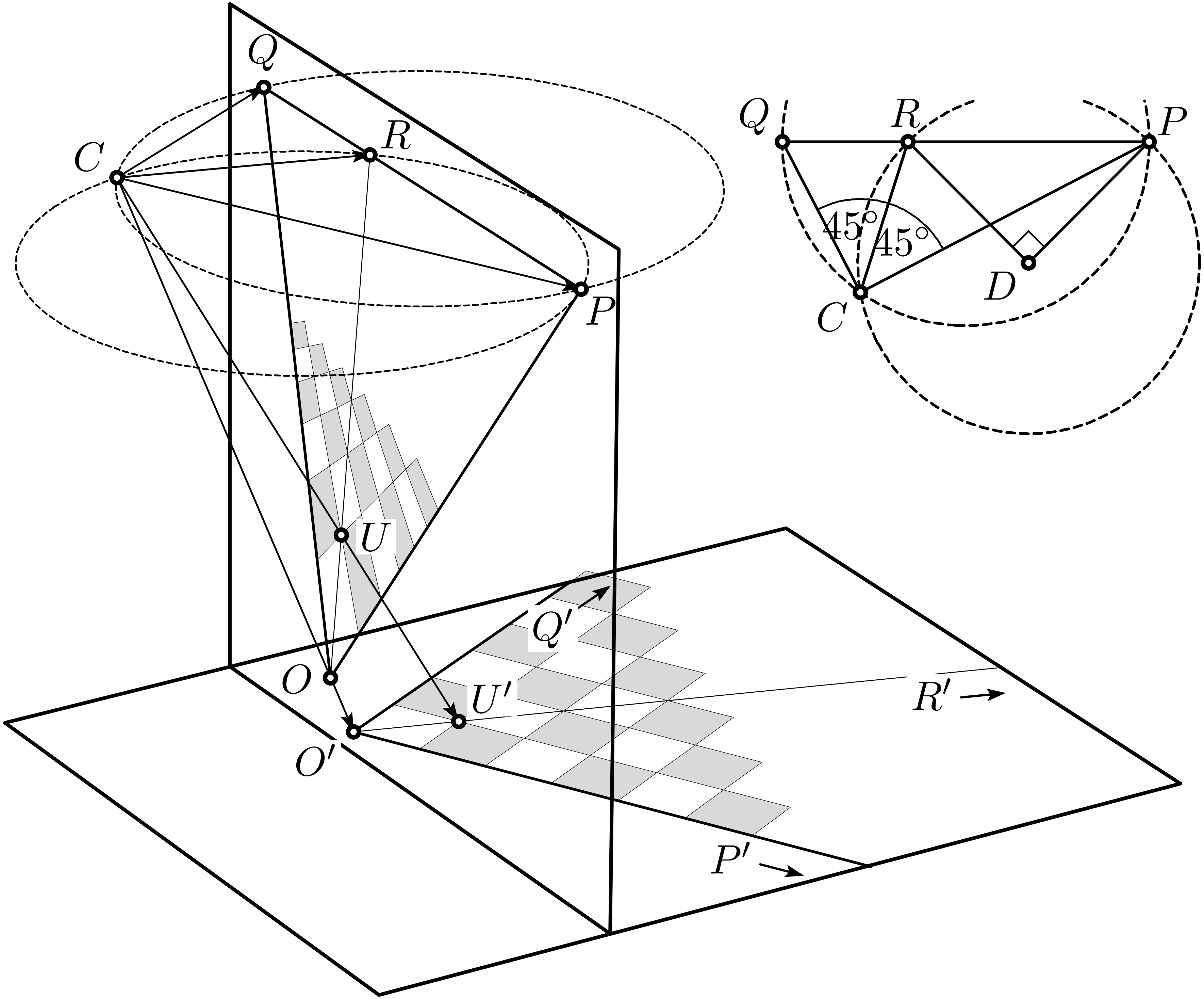
1812 (Russia) – 1822 (publ):



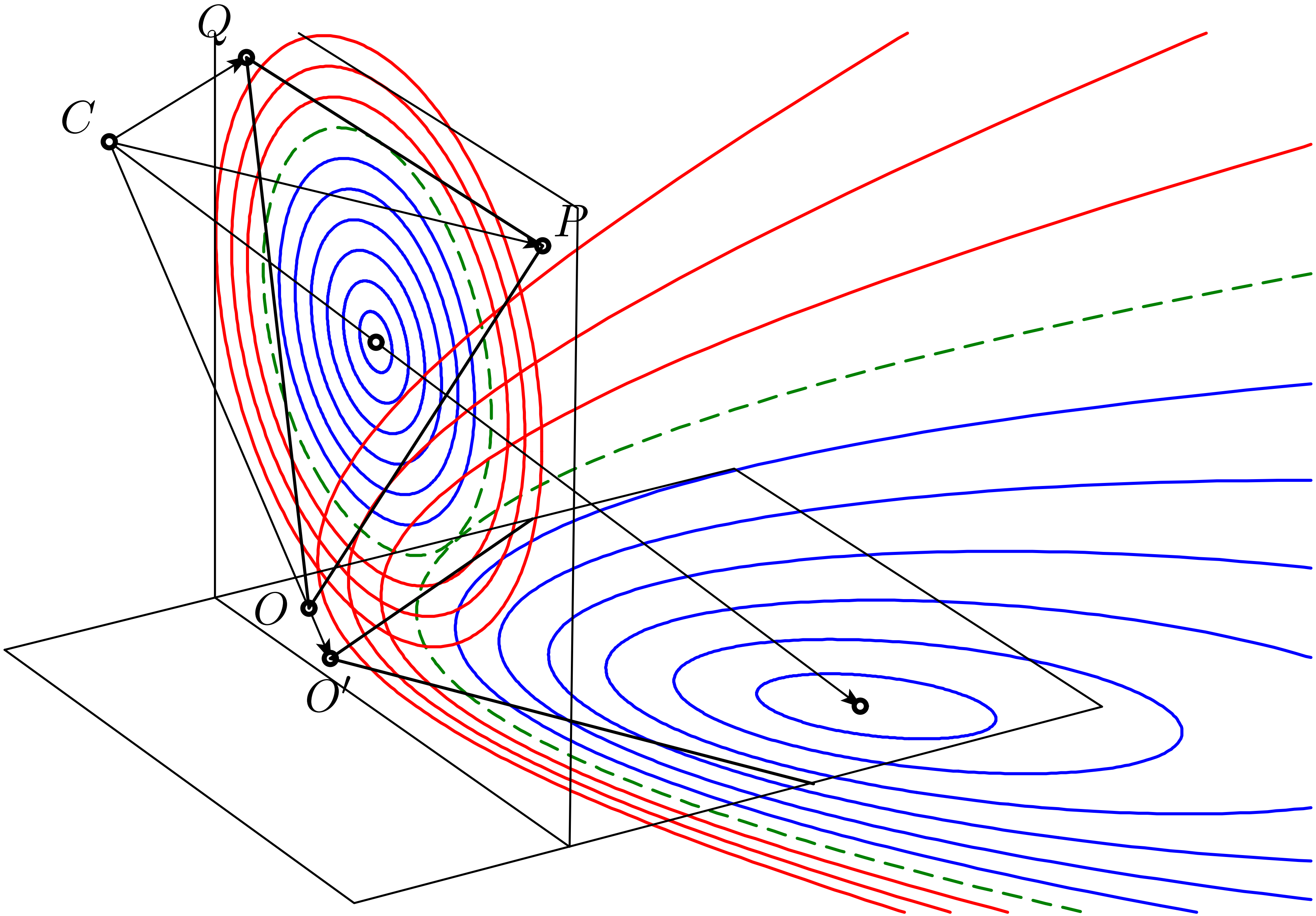
Given triangle OPQ
given "unit" point U,
find central projection
transforming $OPQU$
to orthonormal grid ↓



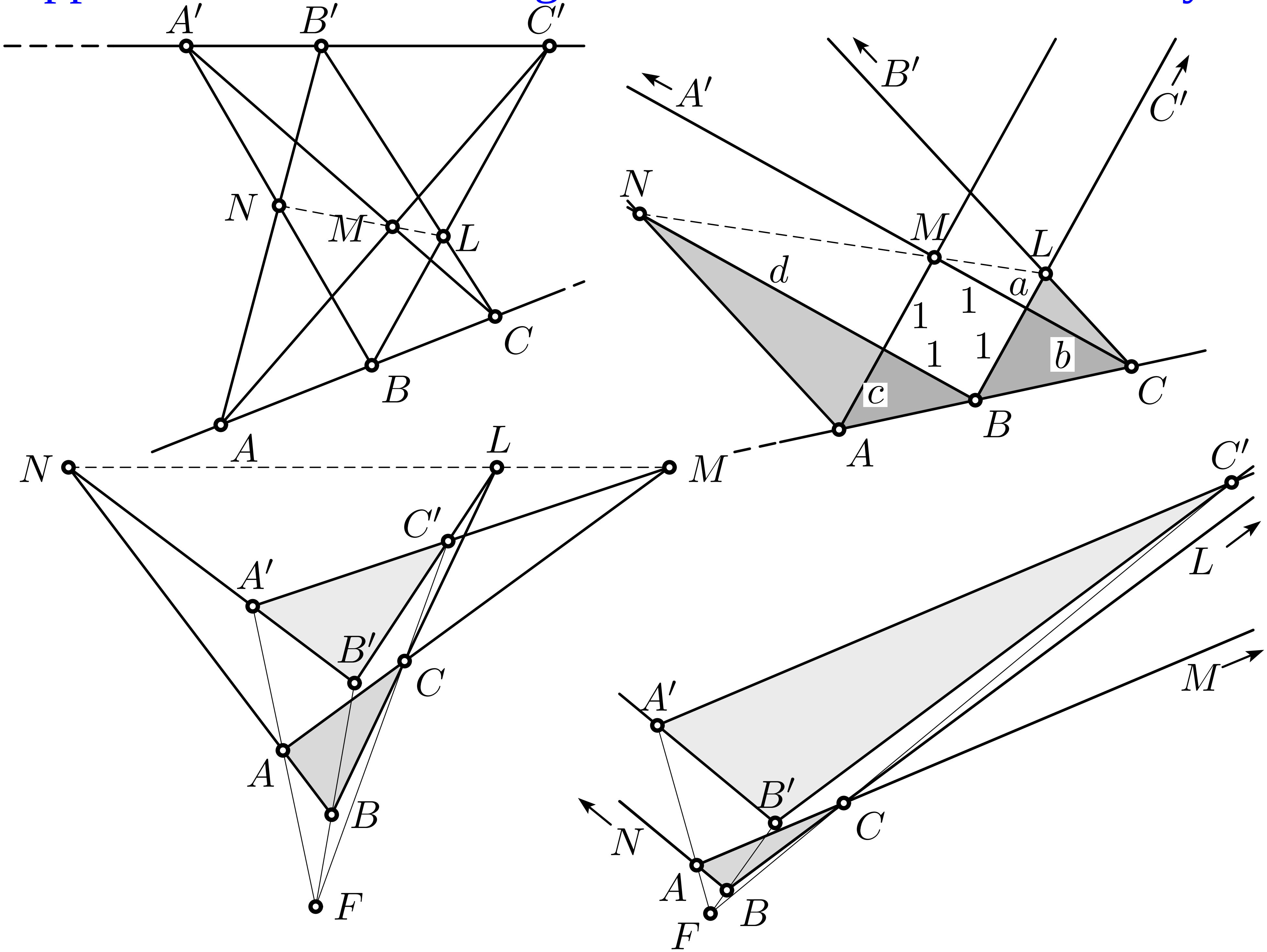
is possible: place QRP horizontally and determine C by Eucl. III.20:



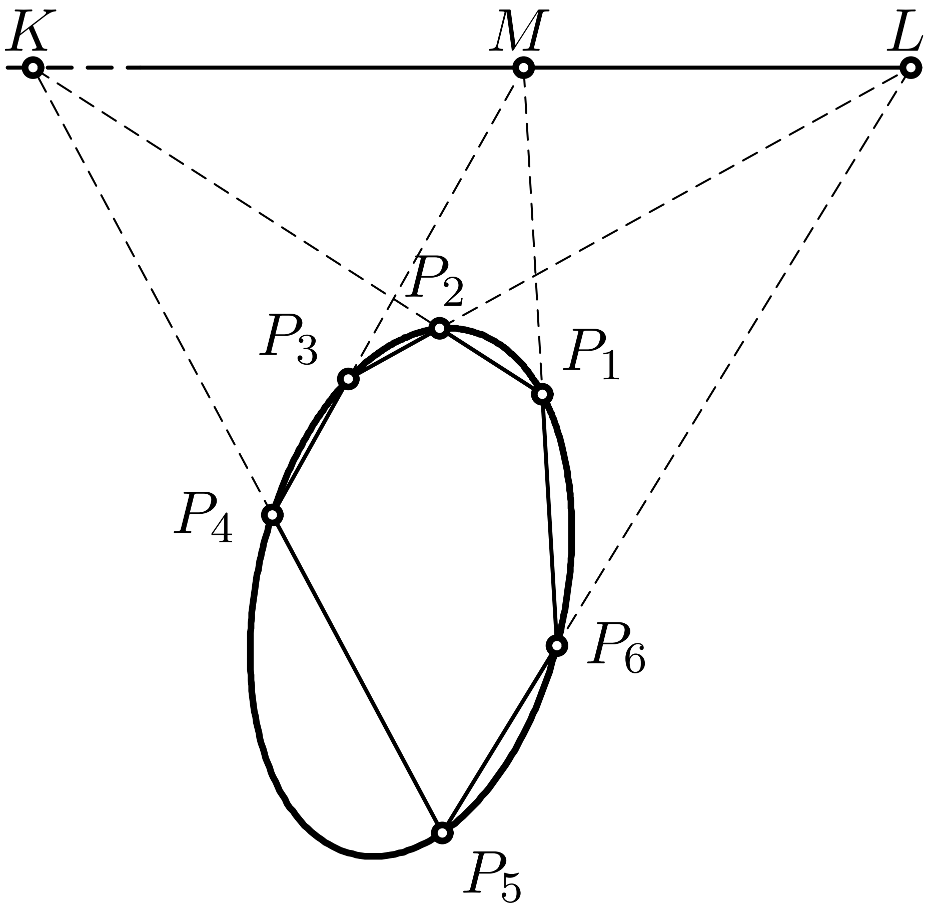
transforms circles to conics and vice-versa:

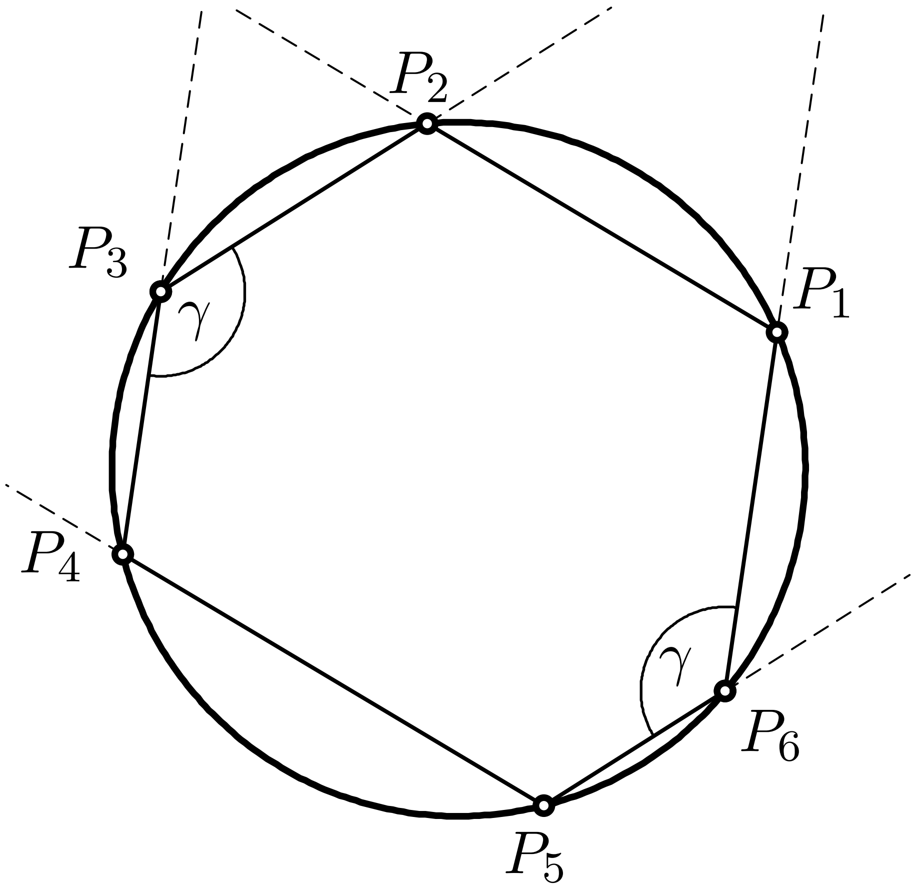


Pappus VII.143 and Desargues' theorem becomes triviality:



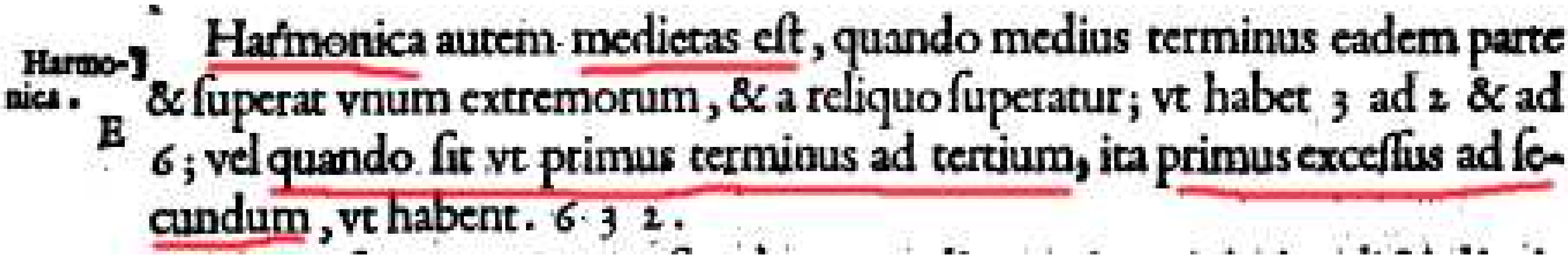
... as well as Pascal's theorem
("hexagramma mysticum" $\Rightarrow$ Eucl. III.21) :





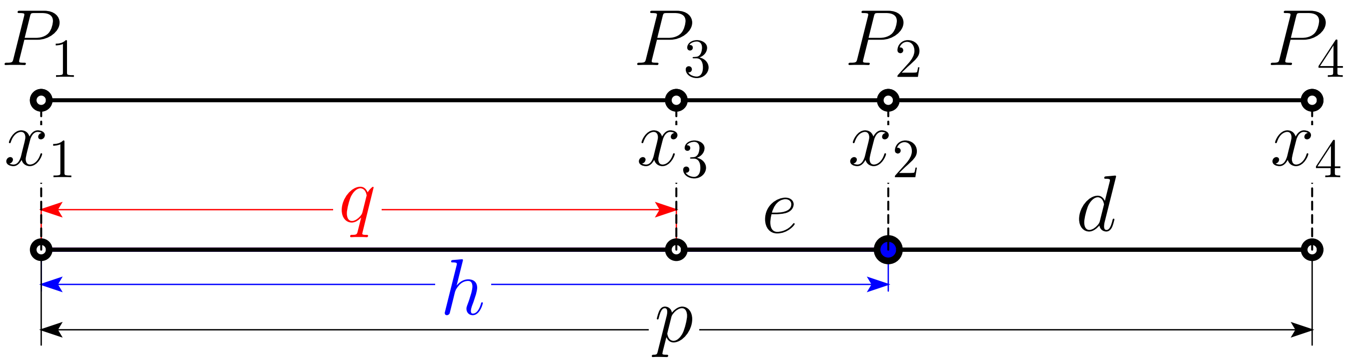
The Cross Ratio:

Remember the def. of harmonic mean :


Harmo-nica. E

Harmonica autem medietas eſt, quando medius terminus eadem parte & ſuperat vnum extremorum, & a reliquo ſuperatur; vt habet 3 ad 2 & ad 6; vel quando ſit vt primus terminus ad tertium, ita primus exceſſus ad ſecundum, vt habent. 6 3 2.

(Pappus, Book III, p. 12 of ed. 1660)



$$\boxed{h = \text{harm. m. of } p, q} \iff \frac{q}{e} = \frac{p}{d} \iff -\frac{x_3 - x_1}{x_3 - x_2} = \frac{x_4 - x_1}{x_4 - x_2}.$$

or if the cross-ratio

$$\text{XR}\,(P_1, P_2, P_3, P_4) = \frac{x_3 - x_1}{x_3 - x_2} : \frac{x_4 - x_1}{x_4 - x_2} = -1\,.$$

The four points P_1, P_2, P_3, P_4 are then called *harmonic*.

J.Steiner, "*Systematische Entwickelung ...*" (240 pages, 1832):

Theorem (Pappus VII.129). The cross-ratio of four points

$$\mathrm{XR}\,(P_1, P_2, P_3, P_4) = \frac{P_1P_3}{P_2P_3} : \frac{P_1P_4}{P_2P_4} = \frac{x_3 - x_1}{x_3 - x_2} : \frac{x_4 - x_1}{x_4 - x_2}$$

is invariant under projective transformations.

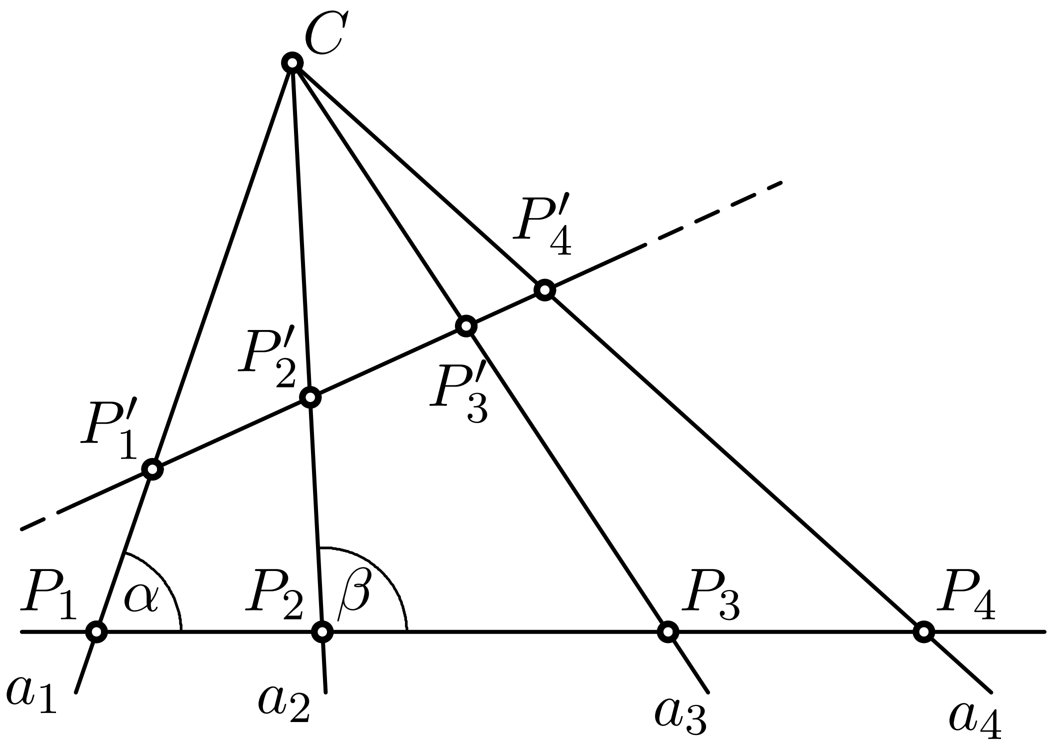
Steiner's Proof:

sine rule:

$$\frac{P_1P_3}{P_2P_3} = \frac{\sin a_1a_3}{\sin a_2a_3} \cdot \frac{\sin\beta}{\sin\alpha}\,,$$

$$\frac{P_1P_4}{P_2P_4} = \frac{\sin a_1a_4}{\sin a_2a_4} \cdot \frac{\sin\beta}{\sin\alpha}\,.$$

$$\Rightarrow \mathrm{XR} \; = \frac{\sin a_1a_3}{\sin a_2a_3} : \frac{\sin a_1a_4}{\sin a_2a_4}\,.$$



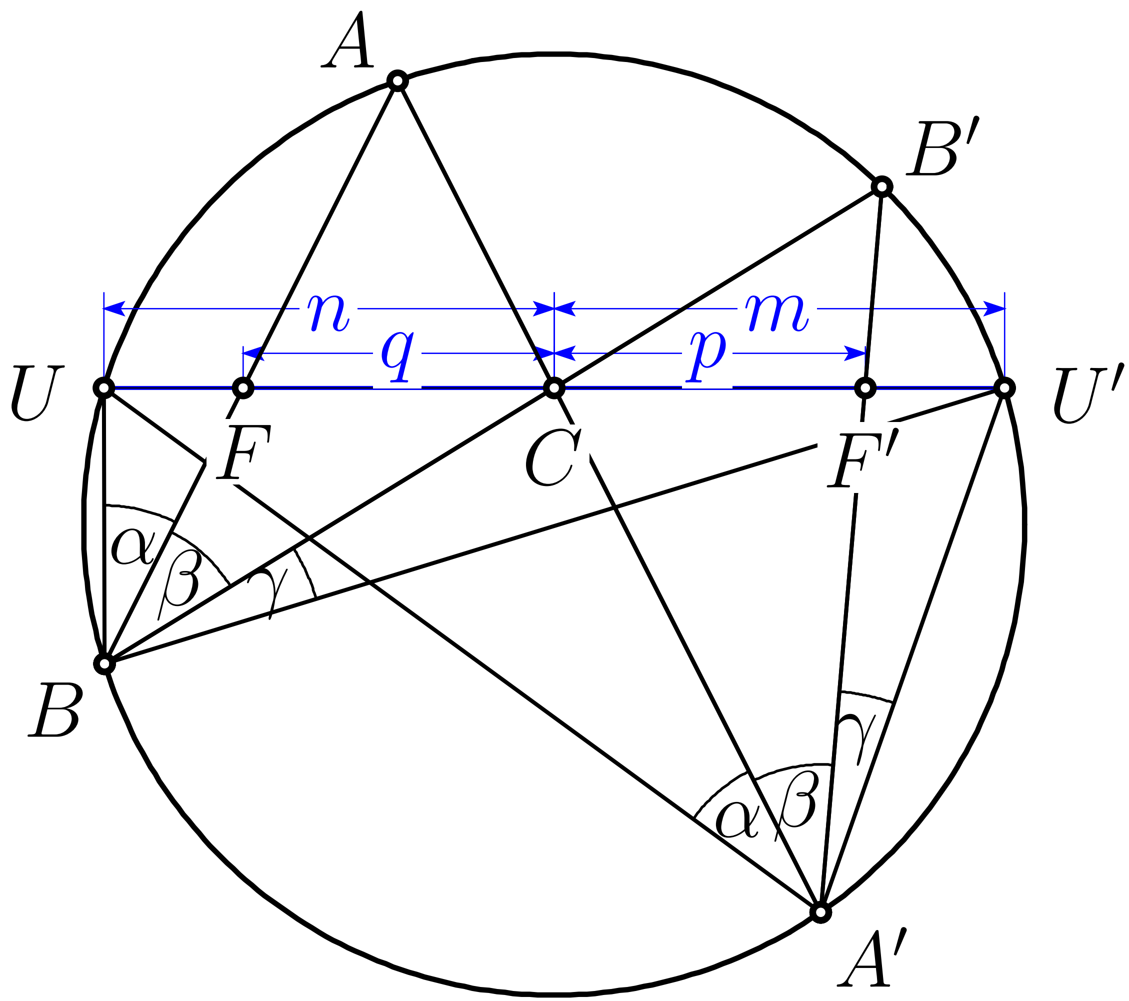
XR depends only on angles at C, $\Rightarrow$ XR (a_1, a_2, a_3, a_4).

Example: The Butterfly. 4 points U, A, B', U' on circle;

(Eucl. III.2: $\alpha = \alpha, \beta = \beta, \gamma = \gamma$ and Thm. Steiner) $\Rightarrow$

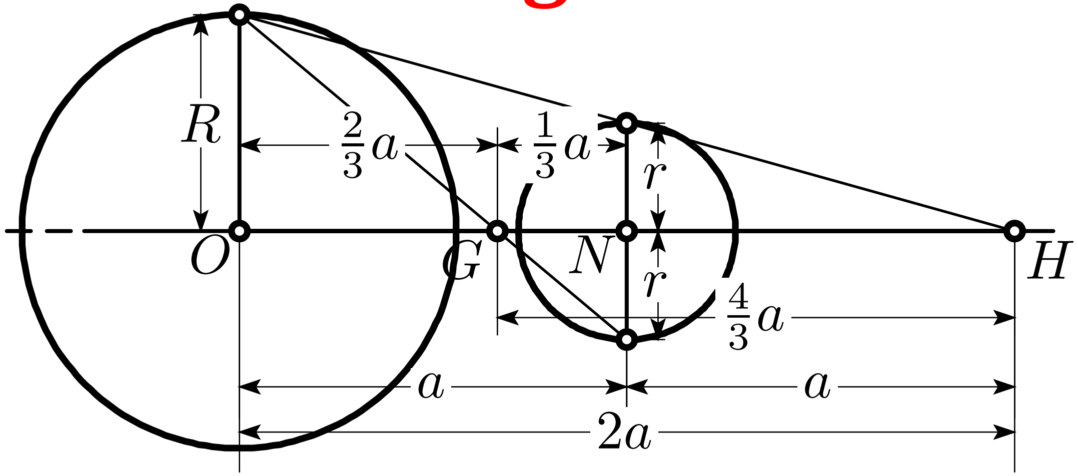
$$\text{XR}\,(UB, AB, B'B, U'B) = \text{XR}\,(UA', AA', B'A', U'A')$$

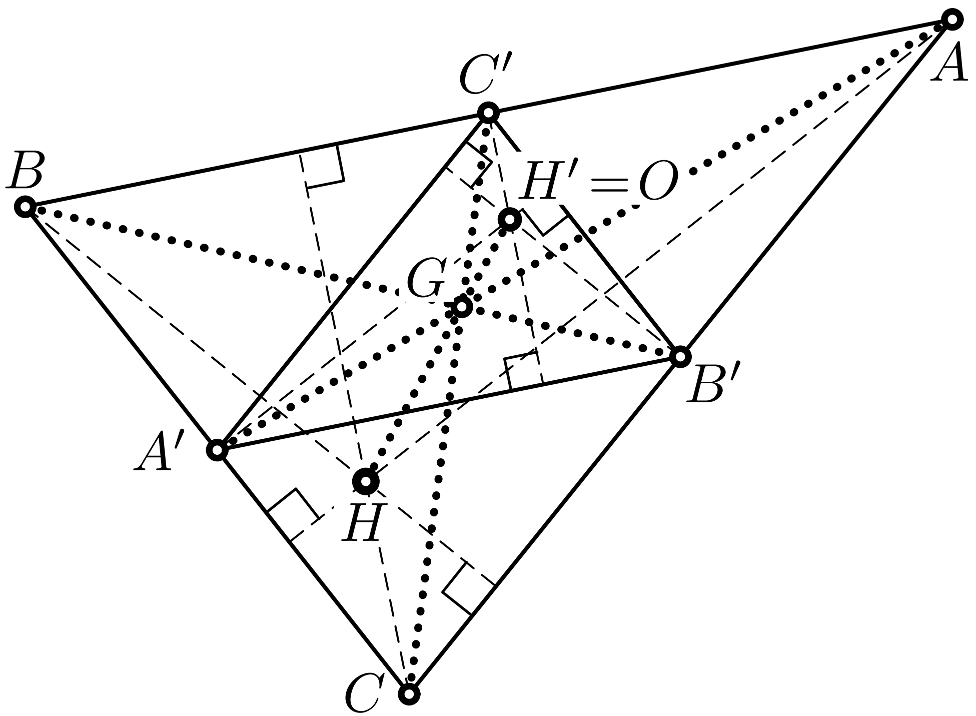
$$\Rightarrow \text{XR}\,(U, F, C, U') = \text{XR}\,(U, C, F', U') \text{ or } \boxed{\frac{1}{m} + \frac{1}{q} = \frac{1}{p} + \frac{1}{n}}.$$

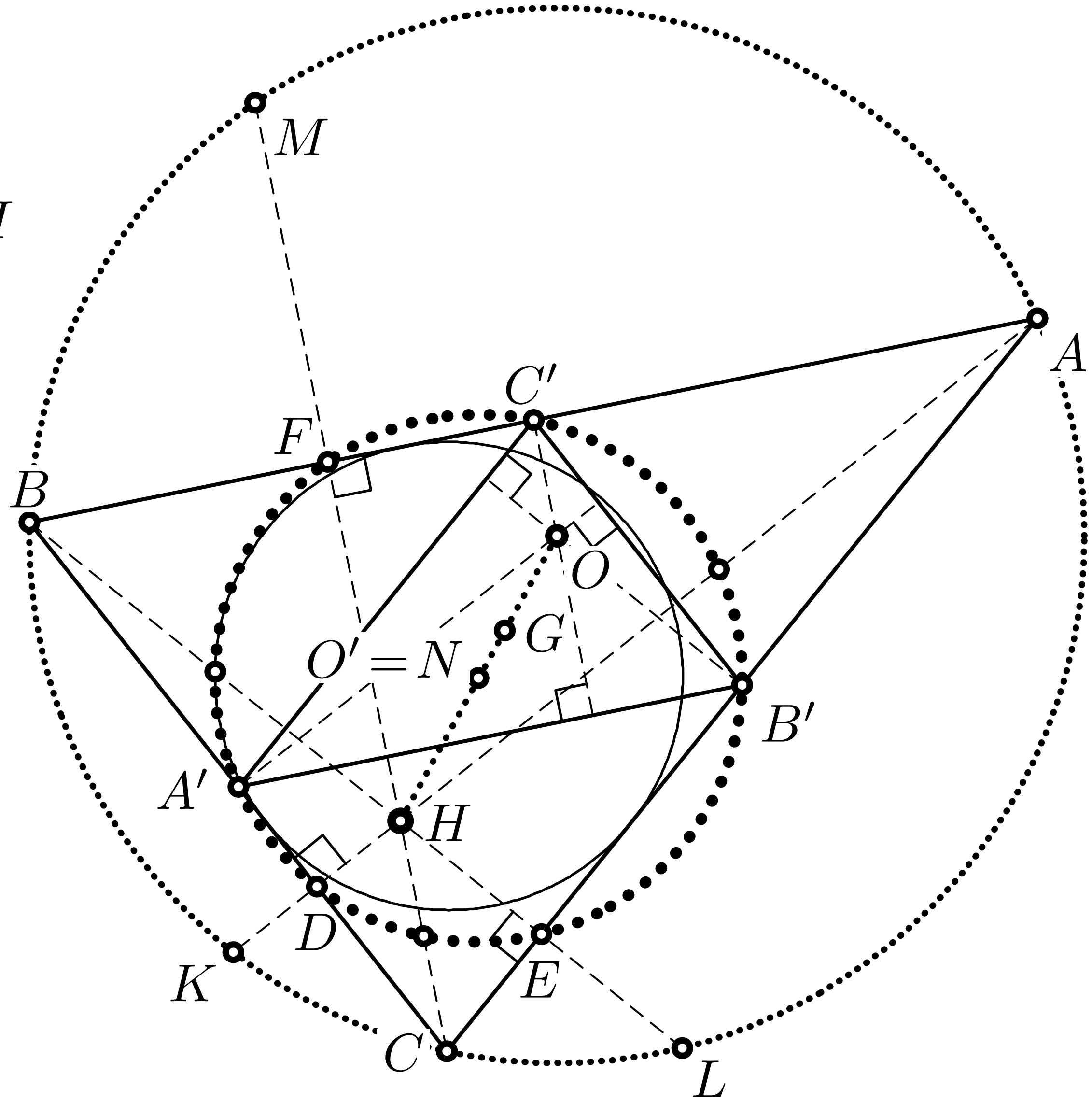


same is true
for any conic
(after projection)
"Steiner's Theorem"

The Triangle. Steiner's view of the Euler line.







$ABC \to A'B'C' =$ medial red., side bisectors $\to$ altitudes,

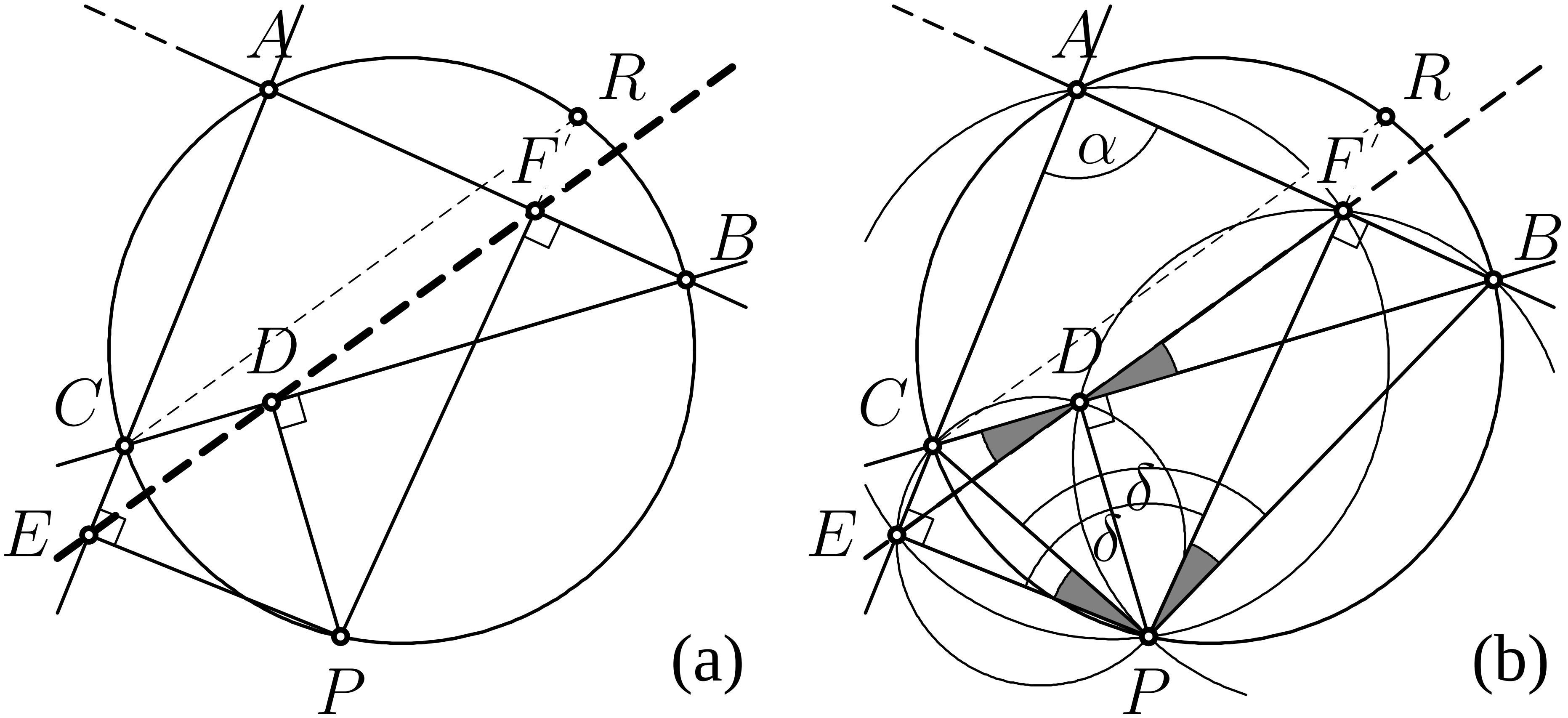
circumcircle $R \to$ nine point circle $r = \frac{R}{2}$,

G inner, H outer sim. center $\Rightarrow$ aligned, $GO = 2HG$, $NO = HN$.

The Simson Line. (F.-J. Servois 1813): *Géométrie pratique*:

“le théorème suivant, qui est, je crois, de Simson” (it was not!!):

P on circumcircle; D, E, F orth, proj. to three sides $\Rightarrow$ aligned.



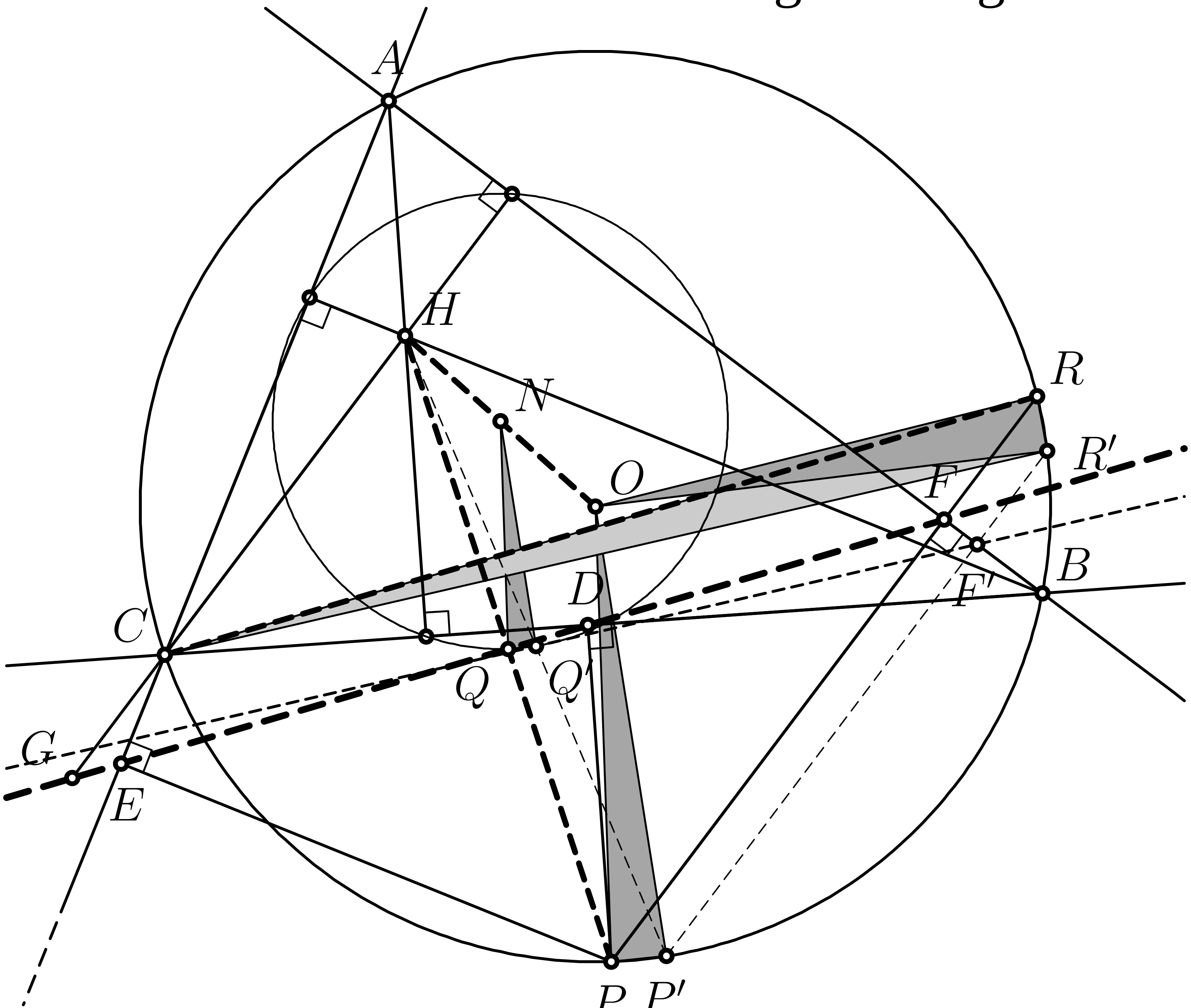
Proof (Servois): • draw circles with diam. PA, PB, PC.;

• Thales circles: D, E, F lie on them;

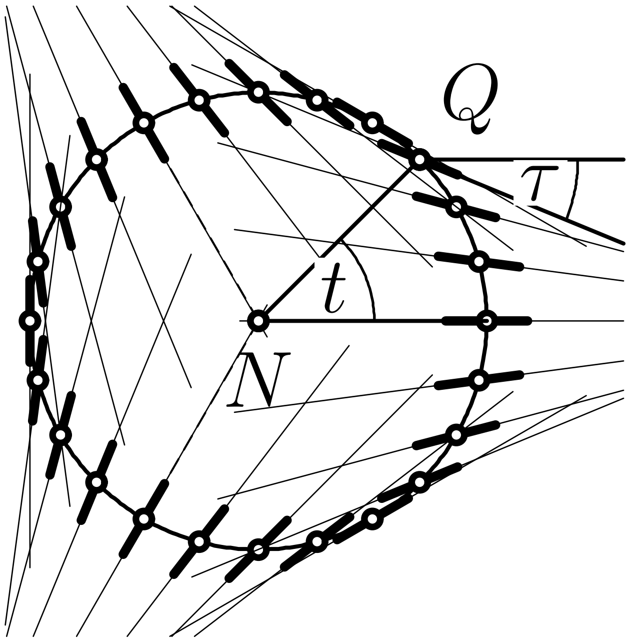
• Eucl. III.22: $\delta = \delta$ (opposite to α);

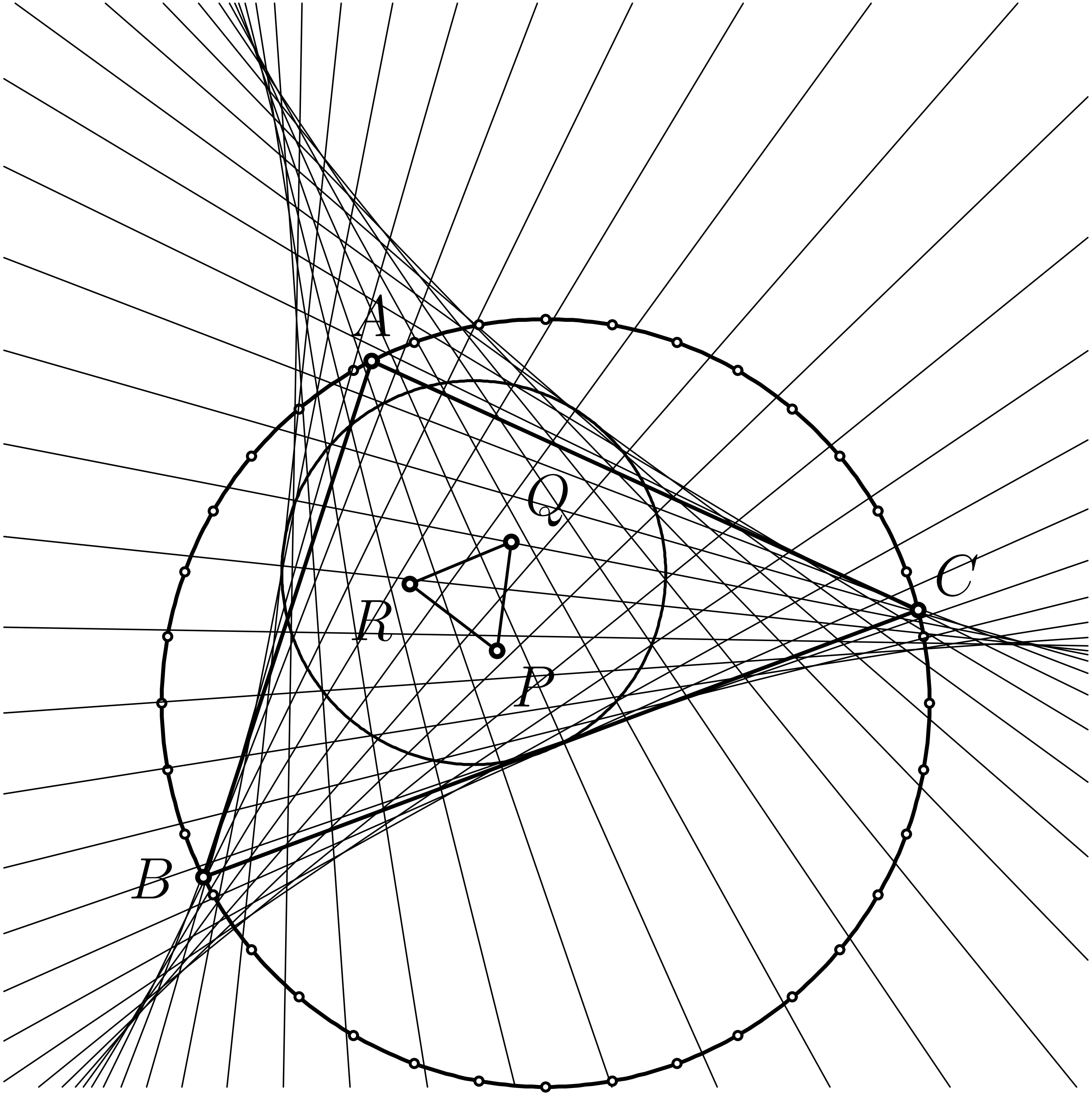
• grey angles at P equal; Eucl. III.21: grey angles at D equal. □

Steiner Deltoid. (Crelle-Borchardt 1857, “... schon beim geradlinigen Dreieck ...”)

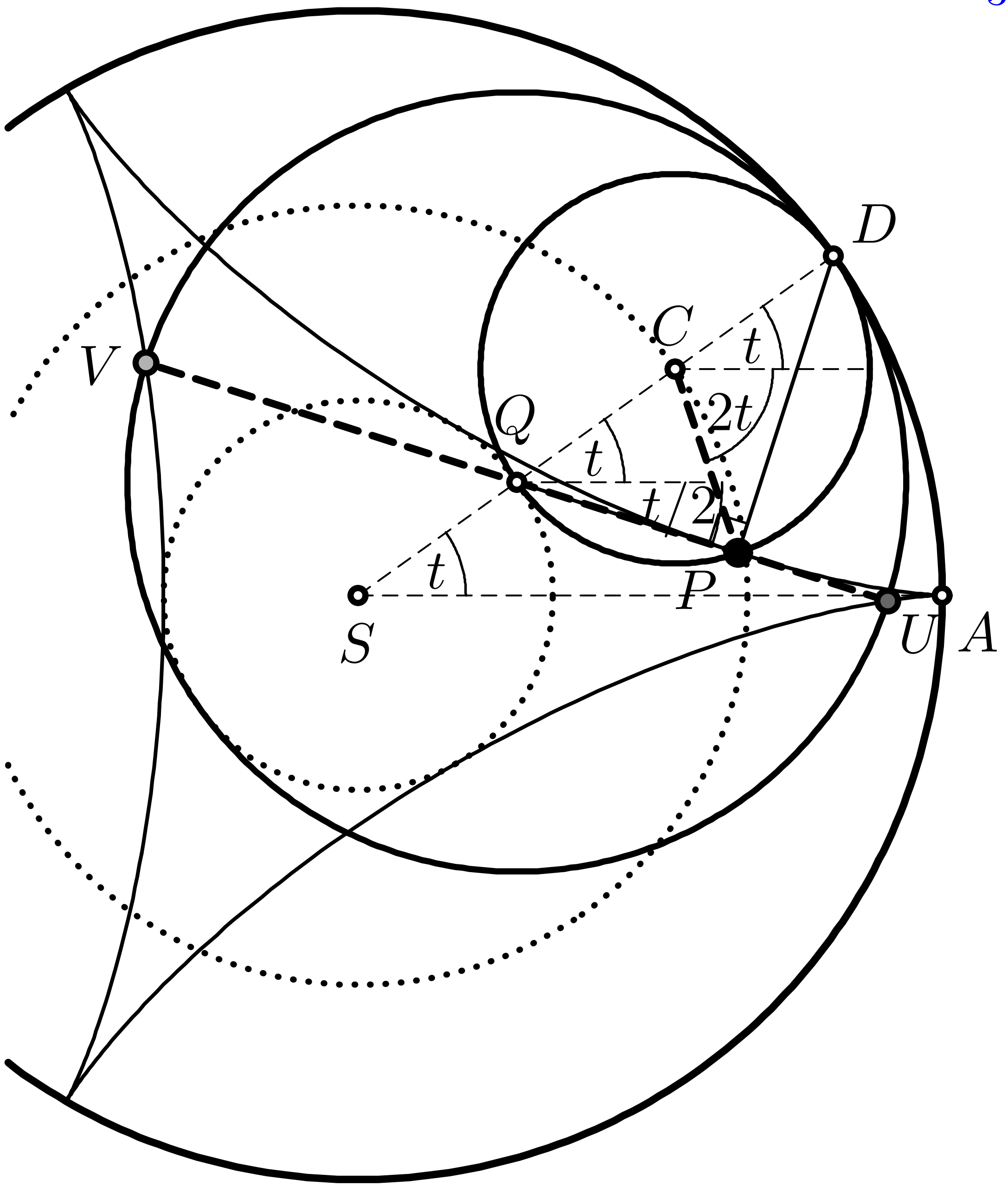


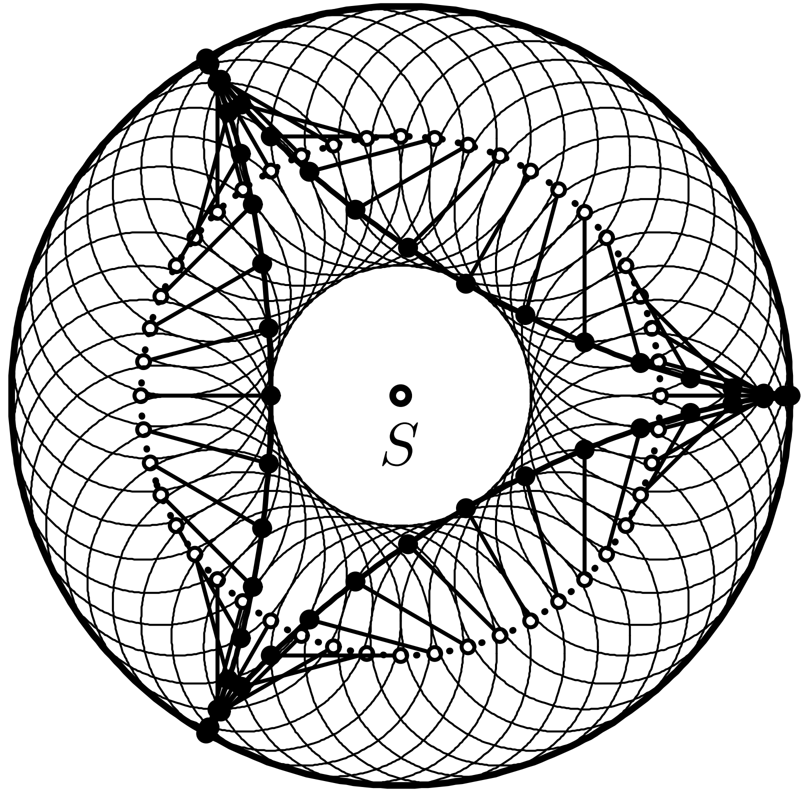
what
happens?

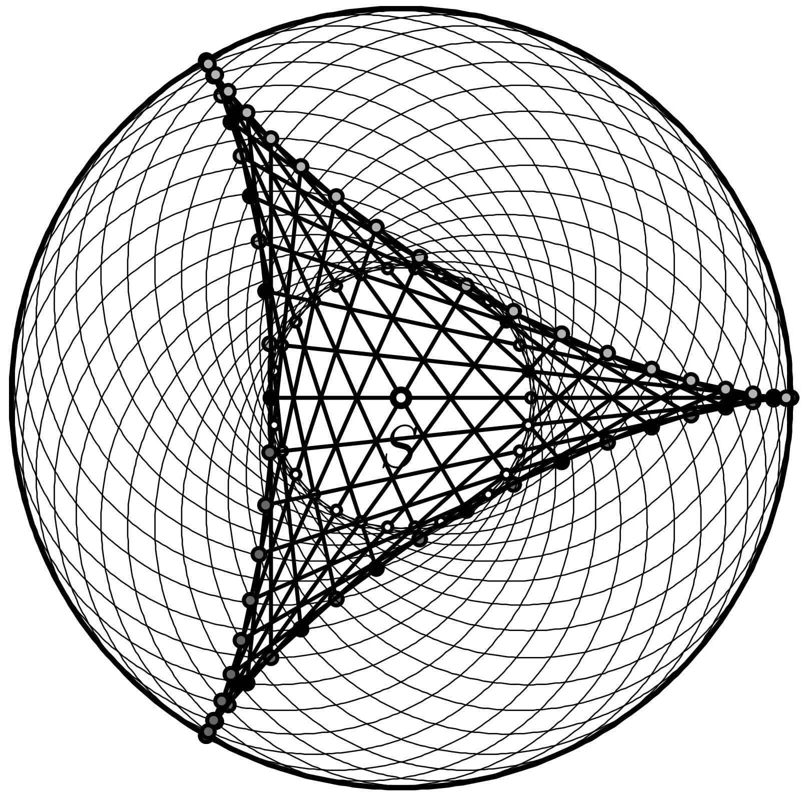




either created by point P on rolling circle $r = \frac{1}{3}$

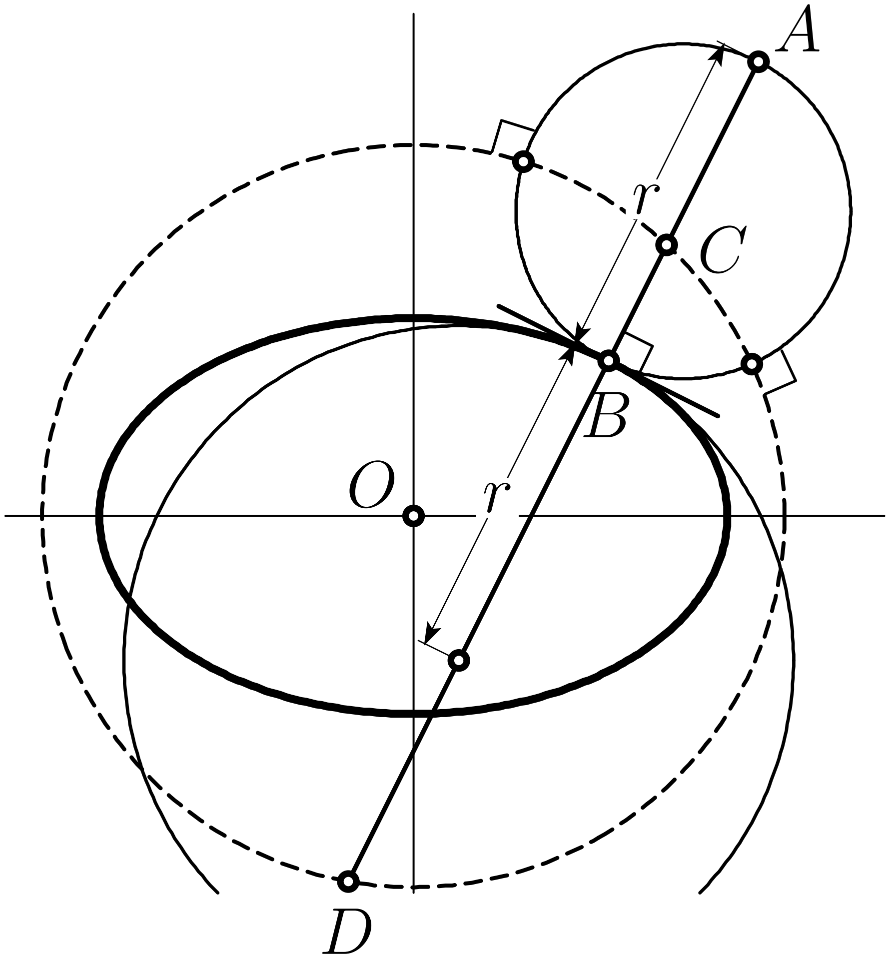






or by diameter VU of rolling circle $r = \frac{2}{3}$

Our last challenge of Steiner for his readers. (Crelle 1846)



r radius of oscul. circ. of ellipse,

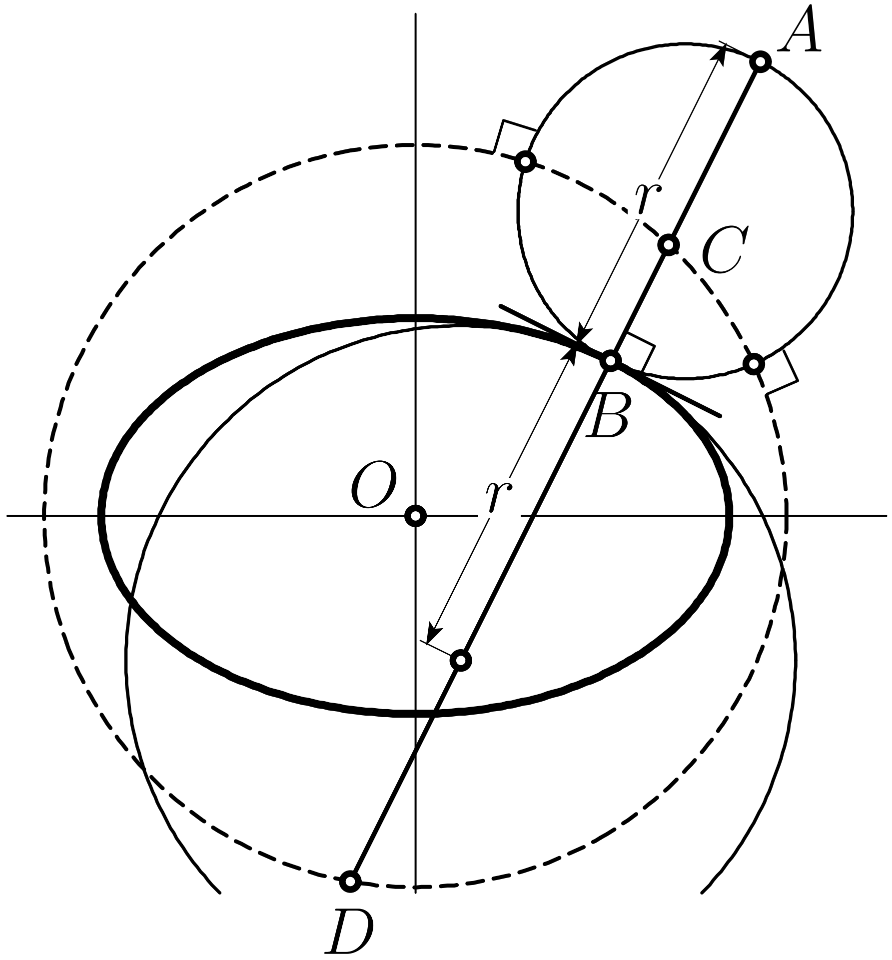
r “nach aussen verlängert...”,

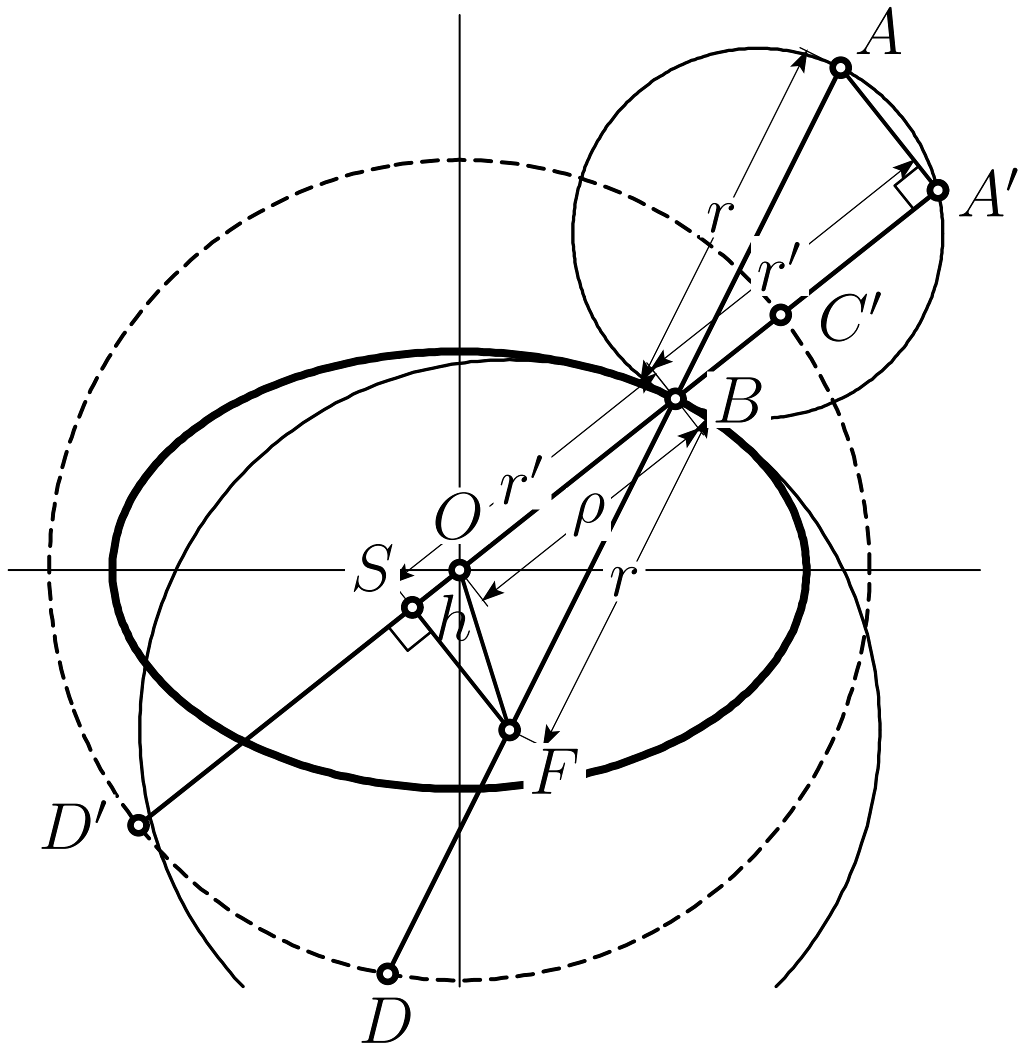
- - - - - circle of radius $\sqrt{a^2 + b^2}$;

show that $ABCD$ are harmonic!

Steiner: no proof, no hint!

Our last challenge of Steiner for his readers. (Crelle 1846)





r radius of oscul. circ. of ellipse,

r "nach aussen verlängert...",

- - - - - circle of radius $\sqrt{a^2+b^2}$;

show that $ABCD$ are harmonic!

Steiner: no proof, no hint!

Proof. $D'OBC'A'$ diam. of Monge circle,

coord. $F = (c^3(a - \frac{b^2}{a}), s^3(b - \frac{a^2}{b}))$

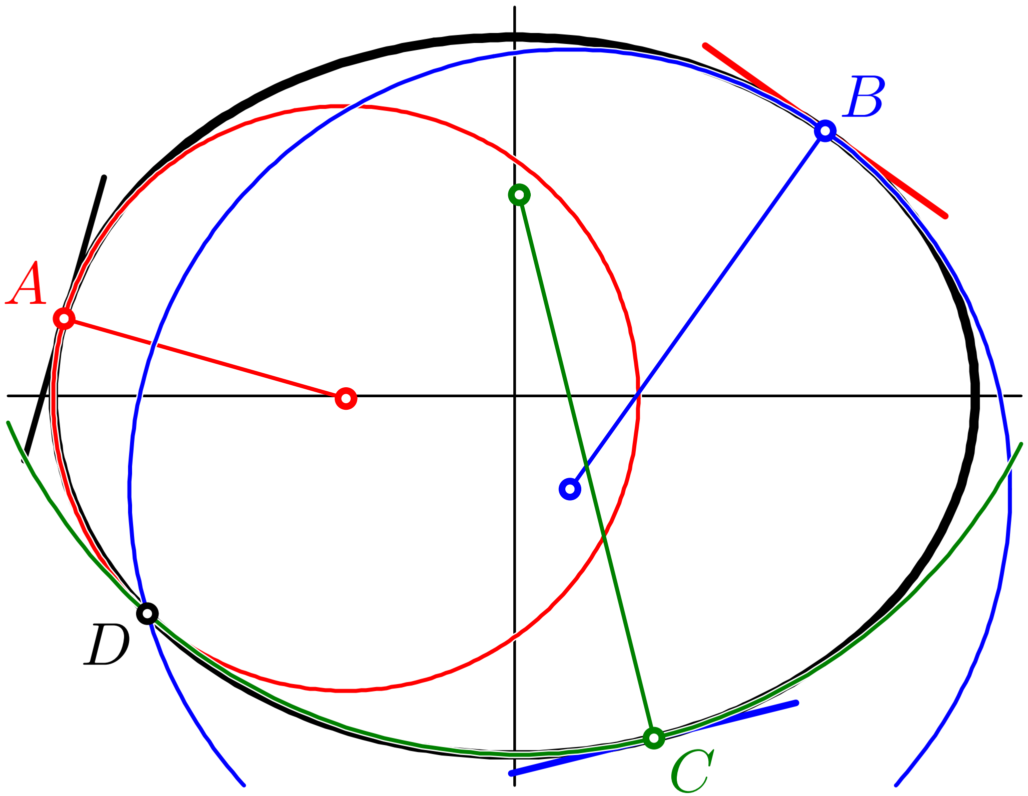
and $B = (ca, sb)$ known since Newton,

have to show $\rho(\rho + r') = a^2 + b^2$.

... and a very last ... Crelle (1846b):

Given point D on ellipse (other than a vertex). Then exist three points A, B, C on the same ellipse, whose osculatory circle passes through D.

Moreover, A, B, C, D are cocyclic.



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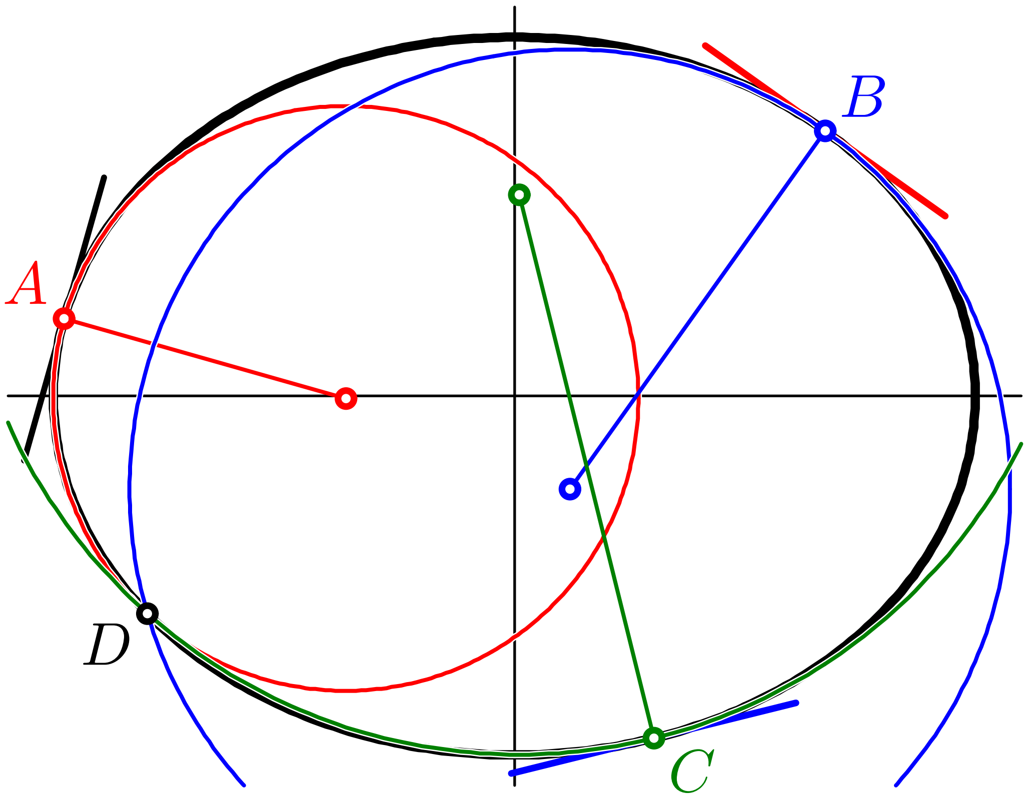
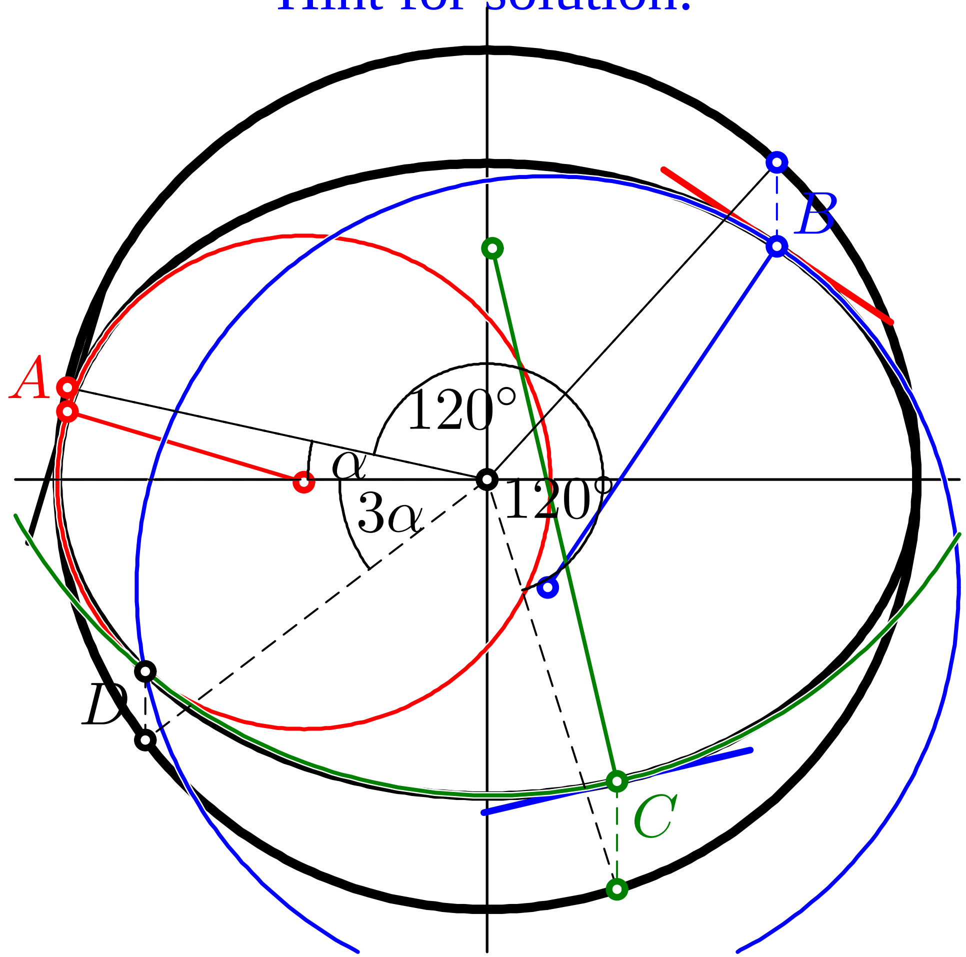
Given point D on ellipse (other than a vertex). Then exist three points A, B, C on the same ellipse, whose osculatory circle passes through D.

Moreover, A, B, C, D are cocyclic.

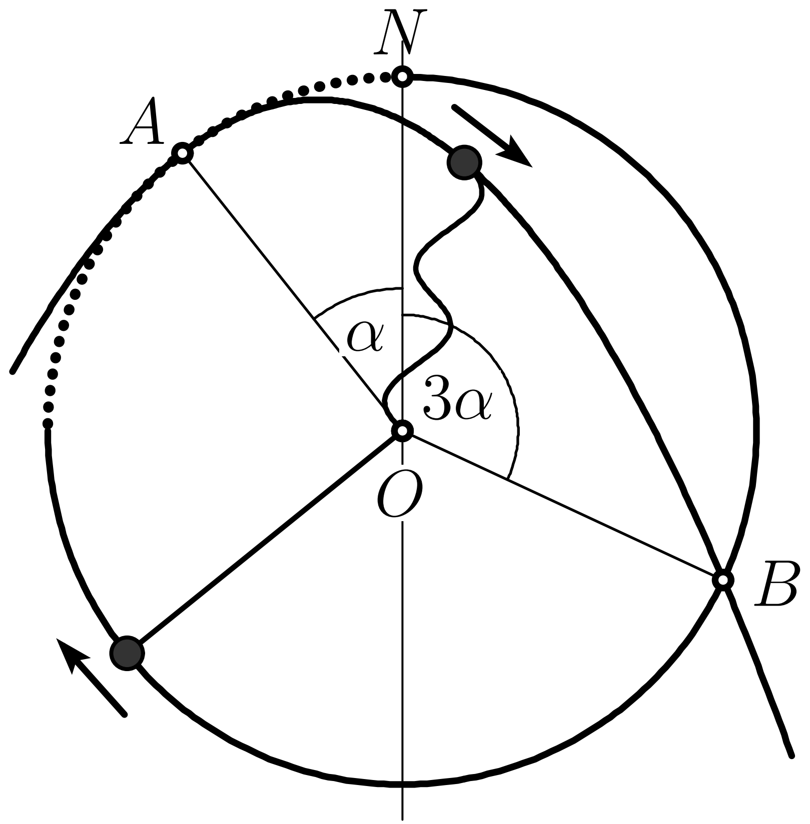


Hint for solution:



Little autobiographic note of the speaker:

When playing around, at the age of 17, with a stone attached to a rope, he discovered that



angle $BON = 3$ angle NOA

which he verified analytically.

He now knows that this is nearly the same as above result of Steiner :-)

Postscriptum (“Schlussbemerkung”):

“In den Abhandlungen ... [Steiners] ...
findet sich eine solche Ueberfülle
ohne Beweis ausgesprochener Sätze, dass ...
auf eine eingehende Prüfung
ihrer Richtigkeit verzichtet werden musste.”

W[eierstrass].

Towards the end of his life,
Steiner (unmarried professor
in Berlin) became rich.
In his last will,
he gave one third of his fortune
to the village of Utzenstorf,
so that **these** children
would get a better school education
than he had.