A NON SPIRALING INTEGRATOR FOR THE LOTKA VOLTERRA EQUATIONS

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Abstract. We consider the solution of the Lotka Volterra System of differential equations in $\mathbb{R}^2$. Almost all numerical methods in use exhibit spiraling, although the exact solution is cyclic. We show, how a simple modification of the Forward Euler method leads to cyclic solutions in the numerical approximation.

1. The Lotka Volterra Equations

We consider a predator species y and its prey x. The Lotka Volterra system of differential equations describes the evolution of x and y by

$$\begin{array}{rcrcrcl} \dot{x} & = & x & - & xy & ; & x(0) = \hat{x}, \\ \dot{y} & = & -y & + & xy & ; & y(0) = \hat{y}. \end{array} \tag{1}$$

The growth rate of the prey population $\dot{x}$ is proportional to the current population x minus the number of predator–prey encounters, proportional to xy. The growth rate of the predator population $\dot{y}$ is proportional to the predator–prey encounters xy minus the current population y. A derivation of this system can be found in Hirsch [2], or in the original paper by Volterra [6].

2. Exact solution

It is well known, that the solution to (1) is cyclic for all initial values $\hat{x}$, $\hat{y}$ in the first quarter plane. The cycles are around the equilibrium point $\bar{x} = 1$, $\bar{y} = 1$, which is obtained by setting the time derivatives on the left hand side of (1) equal to zero. However explicit solutions to this system of equations have only recently been

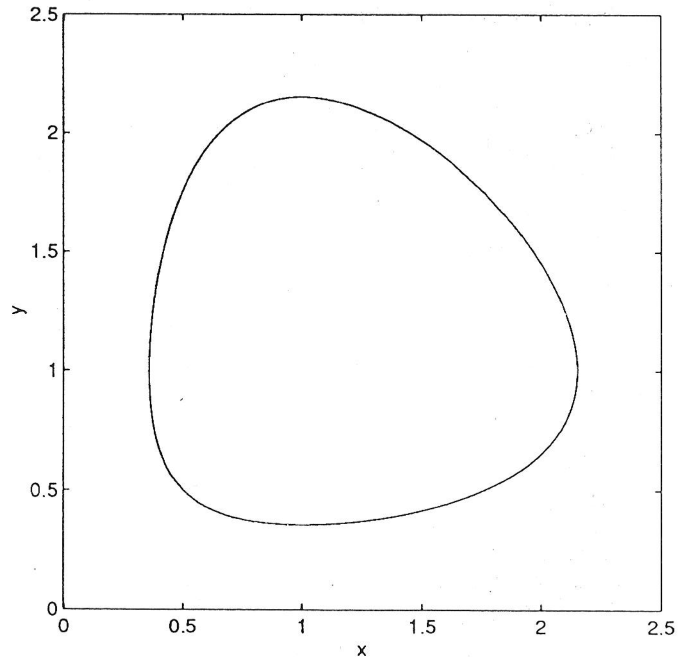

FIGURE 1. Exact solution of the Lotka Volterra System

found by Steiner [5]. The explicit solution is given in parametric form,

$$\begin{aligned} x &= \frac{1}{2}\tau \pm \frac{1}{2}\sqrt{\tau^2 - 4Ce^{\tau}}, \\ y &= \frac{1}{2}\tau \mp \frac{1}{2}\sqrt{\tau^2 - 4Ce^{\tau}}, \end{aligned}$$

where the constant C is given by

$$C = \hat{x}\hat{y}e^{\hat{x}+\hat{y}}.$$

Figure 1 shows the solution for initial values $\hat{x} = 0.5$ and $\hat{y} = 0.5$.

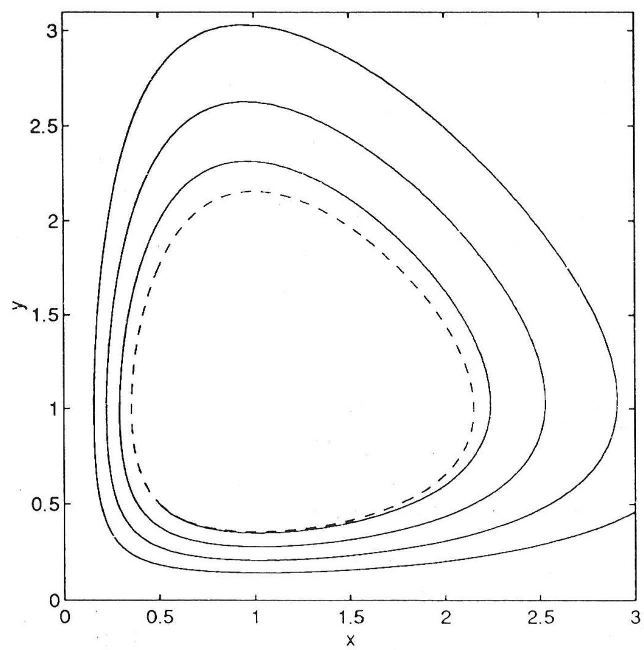

FIGURE 2. Spiraling solution with Forward Euler

3. Forward Euler method

The Forward Euler method discretizes the time derivative in equation (1) by a forward difference. We get the discrete system at time t_n

$$\begin{aligned} \frac{x_{n+1} - x_n}{\Delta t} &= x_n - x_n y_n \quad ; \quad x_0 = \hat{x}, \\ \frac{y_{n+1} - y_n}{\Delta t} &= -y_n + x_n y_n \quad ; \quad y_0 = \hat{y}. \end{aligned} \tag{2}$$

This is an explicit iteration formula. The solution obtained with a time step $\Delta t = 0.1$ and initial values $\hat{x} = 0.5$, $\hat{y} = 0.5$ together with the exact solution is shown in figure 2. The numerical solution obtained by Forward Euler is spiraling, and this is the case for almost all numerical methods one can think of, even for such elaborate techniques like the Runge Kutta method, or Adams Bashforth and

Adams Moulton predictor corrector formulas (cf Golub [1]). Only Kahan's unconventional numerical method is not spiraling [3]. We show in the next section how a simple modification to the Forward Euler approximation leads to a non spiraling method.

4. Modified Forward Euler method

The modification we make is rather simple. We replace x_n in the product term of the second equation of (2) by its newest value x_{n+1} to get the numerical iteration

$$
\begin{array}{rcrcrcl}
\dfrac{x_{n+1}-x_n}{\Delta t} & = & x_n & - & x_n y_n & ; & x_0 = \hat{x}, \\
\dfrac{y_{n+1}-y_n}{\Delta t} & = & -y_n & + & x_{n+1} y_n & ; & y_0 = \hat{y}.
\end{array} \tag{3}
$$

The idea of this change comes from numerical linear algebra, where one calls the iteration in (2) the Jacobi iteration and the modified one in (3) the Gauss Seidel iteration. Figure 3 shows the solution obtained with the modified Forward Euler method with time step $\Delta t = 0.1$ and initial values $\hat{x} = 0.5$, $\hat{y} = 0.5$. The exact solution is plotted as a dashed curve. Note that the modified Forward Euler method is still explicit.

5. Proof of cyclic behavior

Note that (1) can be written as

$$
\begin{array}{rcll}
\dot{x} & = & -xy\dfrac{\partial H}{\partial y}; & x(0) = \hat{x}, \\
\dot{y} & = & xy\dfrac{\partial H}{\partial x}; & y(0) = \hat{y}
\end{array} \tag{4}
$$

with the function

$$H(x,y) = x + y - \ln x - \ln y.$$

Therefore (4) is a non-standard Hamiltonian system. It would be Hamiltonian, if the factor xy was not present in (4). By Liouville's theorem, Hamiltonian systems are area preserving; they preserve the quantity $dx \times dy$. By the KAM theory [4], a numerical integration scheme does not spiral, if it preserves the same quantity $dx \times dy$. In our non-standard Hamiltonian case, there is a similar quantity preserved, as the following theorem shows.

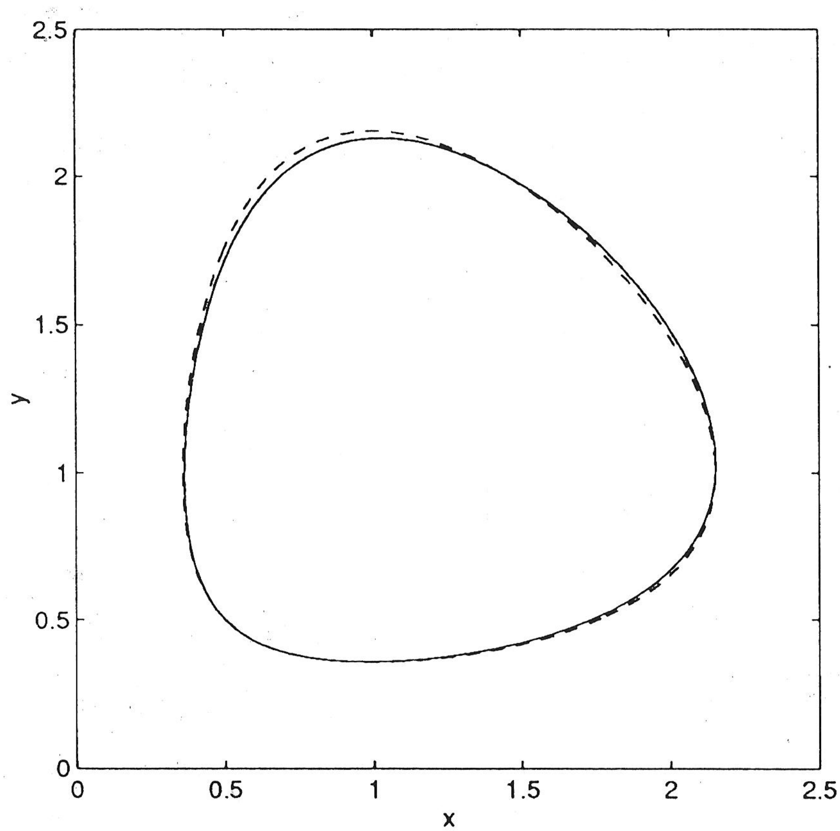

FIGURE 3. Cyclic solution with Modified Euler

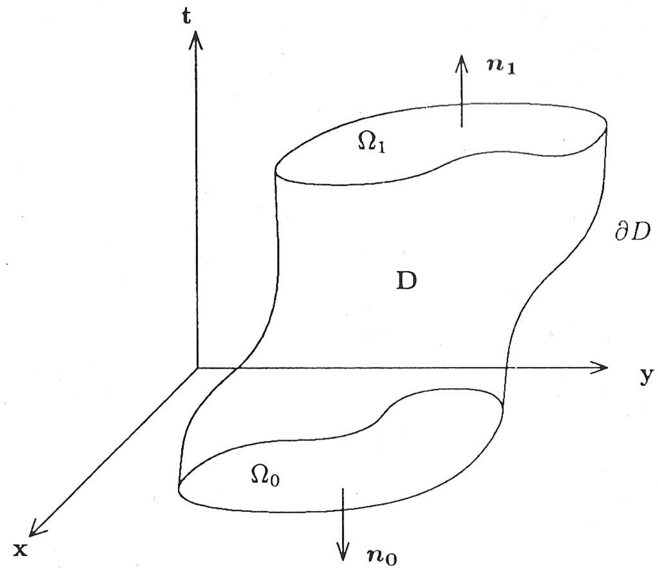

FIGURE 4. Area preserving mapping

Theorem 1. *The Hamiltonian system (4) preserves the weighted area $(dx \times dy)/xy$.*

Proof: Let Ω_0 be a subset of $\mathbb{R}^2$ at time t_0 and Ω_1 the set into which Ω_0 is mapped by (4) at time t_1, as shown in figure 4. Preservation of $(dx \times dy)/xy$ is equivalent to

$$\int_{\Omega_0} \frac{1}{xy}\, dxdy = \int_{\Omega_1} \frac{1}{xy}\, dxdy.$$

We now look at the domain D in x, y, t space with the boundary ∂D given by Ω_0 at t_0, Ω_1 at t_1 and the set of trajectories emerging from the boundary of Ω_0 and ending on the boundary of Ω_1. Consider the vector field

$$\boldsymbol{v} := \frac{1}{xy} \begin{pmatrix} \dot{x} \\ \dot{y} \\ 1 \end{pmatrix}$$

in x, y, t space. Integrating this vector field over the boundary ∂D of D, we obtain

$$\begin{aligned}\int_{\partial D} \boldsymbol{v}\cdot\boldsymbol{n} &= \int_{\Omega_0} \boldsymbol{v}\cdot\boldsymbol{n_0} + \int_{\Omega_1} \boldsymbol{v}\cdot\boldsymbol{n_1} \\ &= \int_{\Omega_0} \frac{1}{xy}\, dxdy - \int_{\Omega_1} \frac{1}{xy}\, dxdy\ ,\end{aligned}$$

where $\boldsymbol{n_0} = (0,0,-1)^T$ and $\boldsymbol{n_1} = (0,0,1)^T$ denote the outward unit normal of Ω_0 and Ω_1. There is no other contribution to the surface integral, because the vector field $\boldsymbol{v}$ is by construction parallel to the trajectories, which form the rest of the boundary ∂D. Applying the divergence theorem to the left hand side of the same equation, we get

$$\begin{aligned}\int_{\partial D} \boldsymbol{v}\cdot\boldsymbol{n} &= \int_D \nabla\cdot\boldsymbol{v} \\ &= \int_D -\frac{\partial H^2}{\partial x\partial y} + \frac{\partial H^2}{\partial x\partial y} + 0 \\ &= 0\ ,\end{aligned}$$

which concludes the proof.

Theorem 2. *The modified Euler scheme (3) preserves the weighted area $(dx \times dy)/xy$. Therefore the numerical solution does not spiral.*

Proof: To simplify notation, we rewrite one step of (3) as

$$\begin{aligned}X &= \Delta tx + x - \Delta txy \\ Y &= -\Delta ty + y + \Delta tXy,\end{aligned}$$

where we have set $X := x_{n+1}$, $Y := y_{n+1}$, $x := x_n$ and $y := y_n$, and solved for the unknowns X and Y. Taking derivatives of both sides, we get

$$\begin{aligned}dX &= \Delta tdx + dx - \Delta tdxy - \Delta txdy \\ dY &= -\Delta tdy + dy + \Delta tdXy + \Delta tXdy.\end{aligned}$$

Now we can compute the cross product $dX \times dY$. We obtain, after some manipulation, observing that $dX \times dX = 0$ and $dY \times dY = 0$,

$$\begin{aligned} dX \times dY &= dx \times dy\Big(\Delta t^3(x - 2xy + xy^2) \\ &\quad +\Delta t^2(-1 + 2x + y - 2xy) + \Delta t(x - y) + 1\Big). \end{aligned}$$

But the product of X and Y is

$$\begin{aligned} XY &= xy\Big(\Delta t^3(x - 2xy + xy^2) + \Delta t^2(-1 + 2x + y - 2xy) \\ &\quad +\Delta t(x - y) + 1\Big), \end{aligned}$$

and therefore, we have

$$\frac{1}{XY} dX \times dY = \frac{1}{xy} dx \times dy$$

and our scheme preserves the weighted area.

References

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