Birth of Scientific Computations

Martin Gander and Gerhard Wanner, Genève

L. Euler

W. Ritz

B.G. Galerkin

Brissago, 30 sept. 2011.

“c’est des calculs monstrueux . . . heureusement on a des ordinateurs . . .”

(R. Durrer, giovedi, ora 9:44)

“ . . . heureusement on a des bonnes méthodes numériques . . .”

(addendum G. Wanner, venerdi, ora 10:34)

Euler (E122, 1747): **Differential Eqs. for Mechanics.**

$$\text{I. } \frac{2ddx}{dt^2} = \frac{X}{M}; \quad \text{II. } \frac{2ddy}{dt^2} = \frac{Y}{M}; \quad \text{III. } \frac{2ddz}{dt^2} = \frac{Z}{M}$$

"While physicists call these "Newton's equations",
they occur nowhere in the work of Newton
or of anyone else prior to 1747."

"... such is the universal ignorance
of the true history of mechanics."

(C. Truesdell, *Essays in the History of Mechanics*, 1968)

Euler (E122, 1747): Differential Eqs. for Mechanics.

$$\text{I. } \frac{2\,ddx}{dt^2} = \frac{X}{M}; \quad \text{II. } \frac{2\,ddy}{dt^2} = \frac{Y}{M}; \quad \text{III. } \frac{2\,ddz}{dt^2} = \frac{Z}{M}$$

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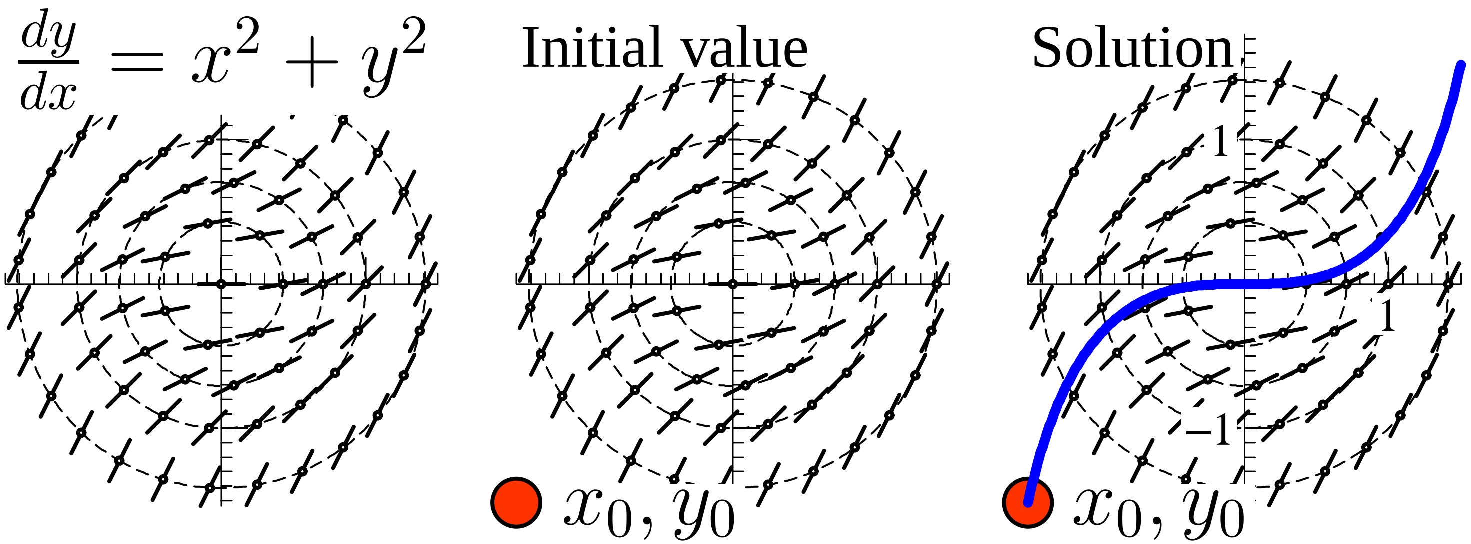
Easier:

First order eq.

$$\frac{dy}{dx} = V(x,y)$$



"Euler's Method" (E342), Inst. Calc. Integralis 1768, §650:

$b' = b + A(a' - a)$

$b'' = b' + A'(a'' - a')$

$b''' = b'' + A''(a''' - a'')$

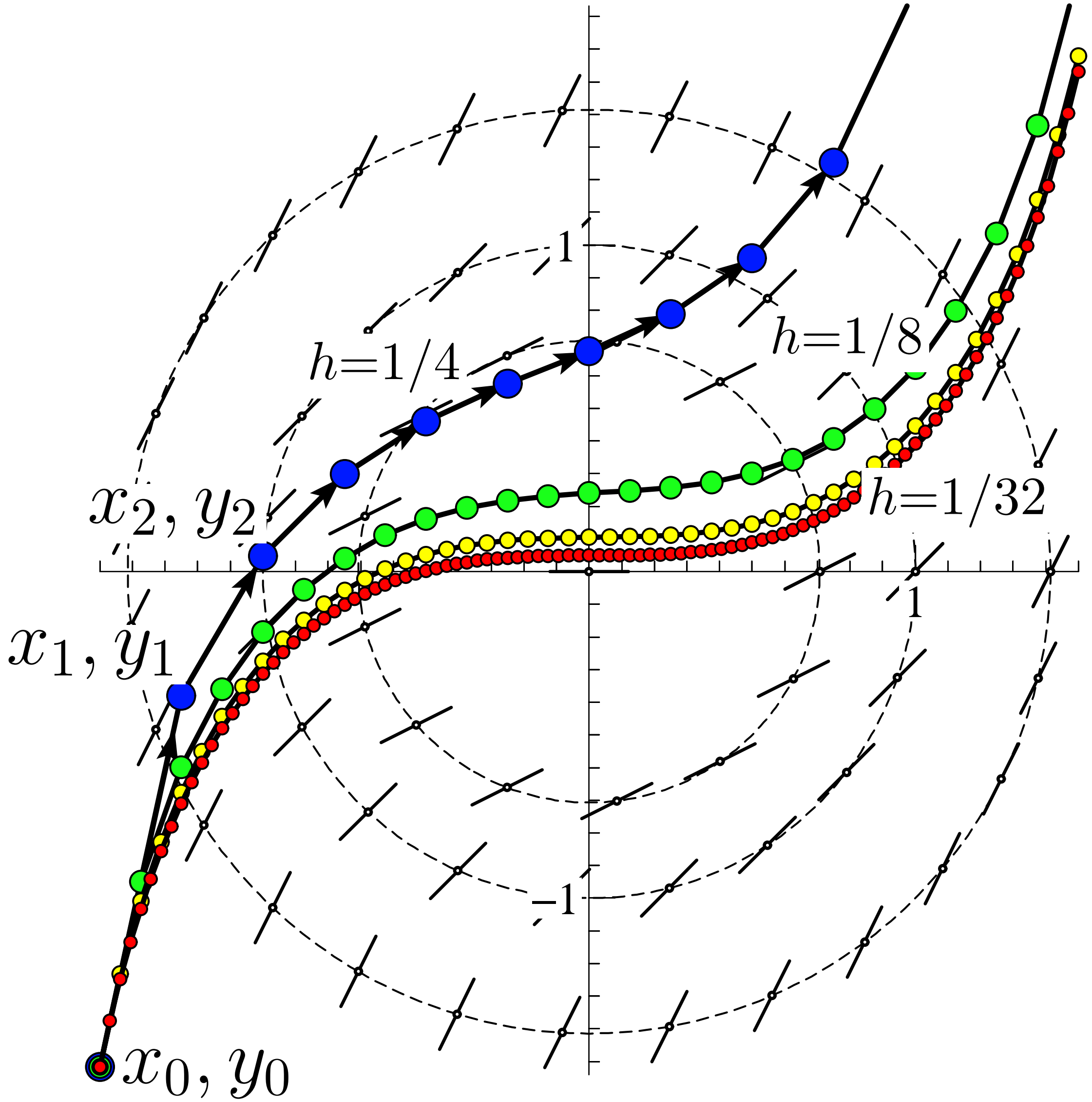
Ipsius	valores successiui
x	a ; a' ; a'' ; a''' ; a^{IV} ; . . . . $'x$; x
y	b ; b' ; b'' ; b''' ; b^{IV} ; . . . . $'y$; y
V	A ; A' ; A'' ; A''' ; A^{IV} ; . . . . $'V$; V.

$$\frac{dy}{dx} = V(x, y)$$

$$x_{n+1} = x_n + h,$$

$$y_{n+1} = y_n + hV(x_n, y_n)$$

(Preuve de convergence: Cauchy 1824)



Higher Order Method (Euler E342), ICI 1768, §656:

$$y = b + \frac{(x-a)\,db}{da} + \frac{(x-a)^2\,ddb}{1\cdot 2\,da^2} + \frac{(x-a)^3\,d^3b}{1\cdot 2\cdot 3\,da^3} + \text{etc.}$$

$$\frac{ddy}{dx^2} = \left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right)$$

$$\frac{d^3y}{dx^3} = \left(\frac{ddV}{dx^2}\right) + \left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right) + 2V\left(\frac{ddV}{dx\,dy}\right) + V\left(\frac{dV}{dy}\right)^2 + VV\left(\frac{ddV}{dy^2}\right).$$

Differentiate for higher derivatives Ex.: $y' = x^2 + y^2$:

Exemplum 2.

662. *Aequationis differentialis* $\partial y = \partial x\,(xx + yy)$ *integrale completum proxime investigare.*

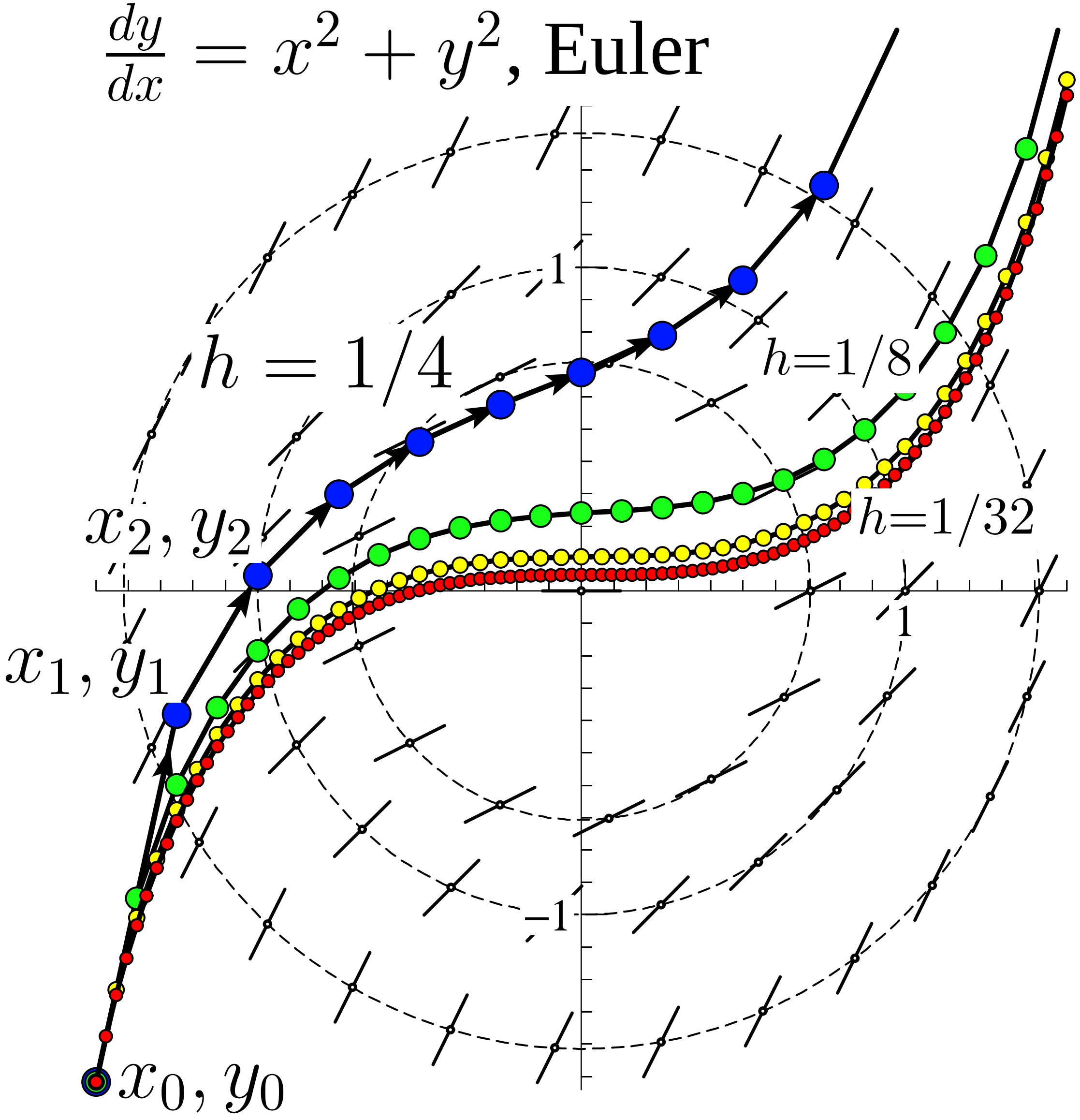
Cum hic sit $\frac{\partial y}{\partial x} = V = xx + yy$, erit continuo differentiando

$$\frac{\partial\partial y}{\partial x^2} = 2x + 2xxy + 2y^3 \text{ et}$$

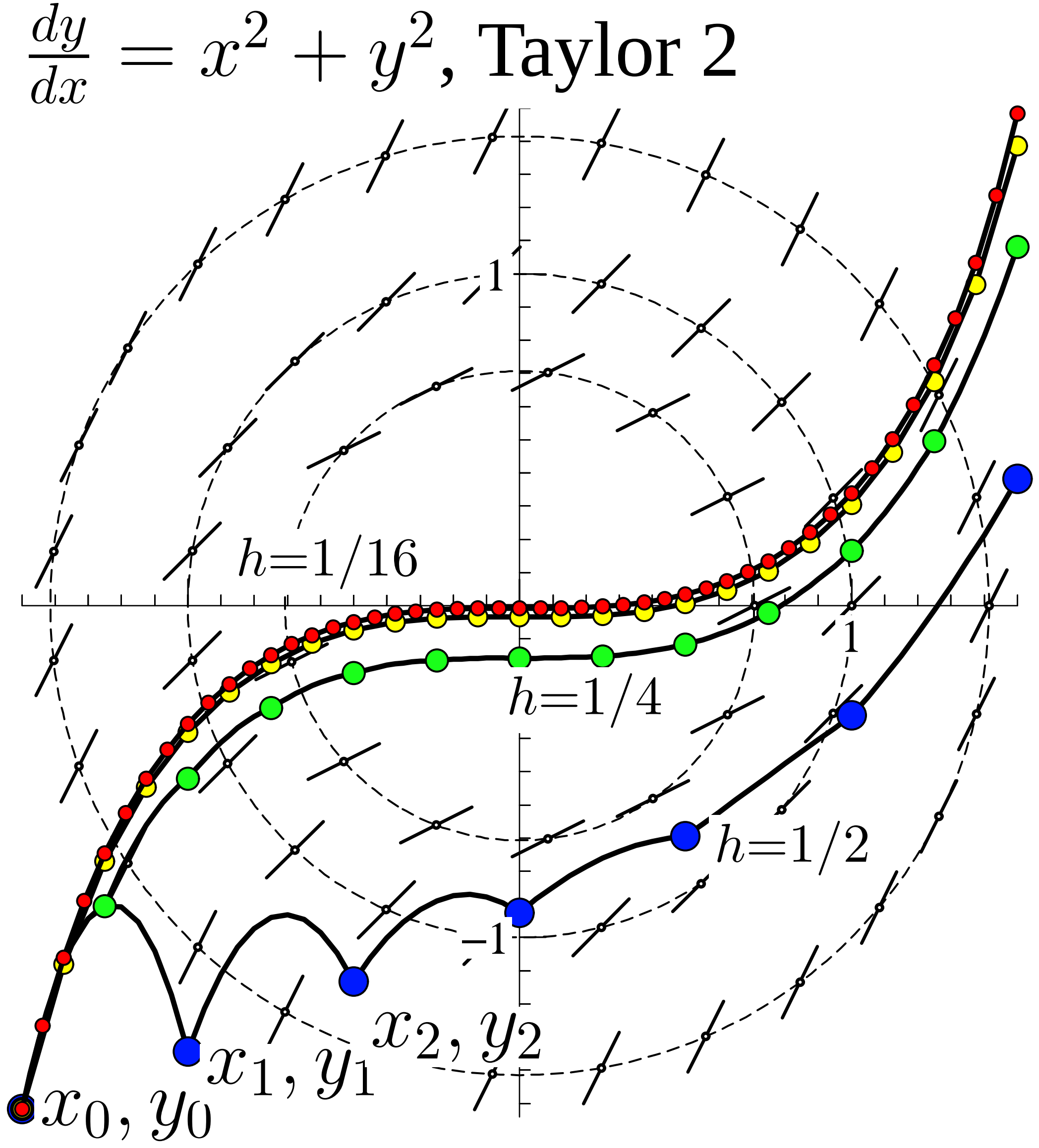
$$\frac{\partial^3 y}{\partial x^3} = 2 + 4xy + 2x^4 + 8xxyy + 6y^4$$

$$\frac{\partial^4 y}{\partial x^4} = 4y + 12x^3 + 20xyy + 16x^4y + 40xxy^3 + 24y^5$$

$$\frac{\partial^5 y}{\partial x^5} = 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 136x^4y^2 + 240x^2y^4 + 120y^6.$$

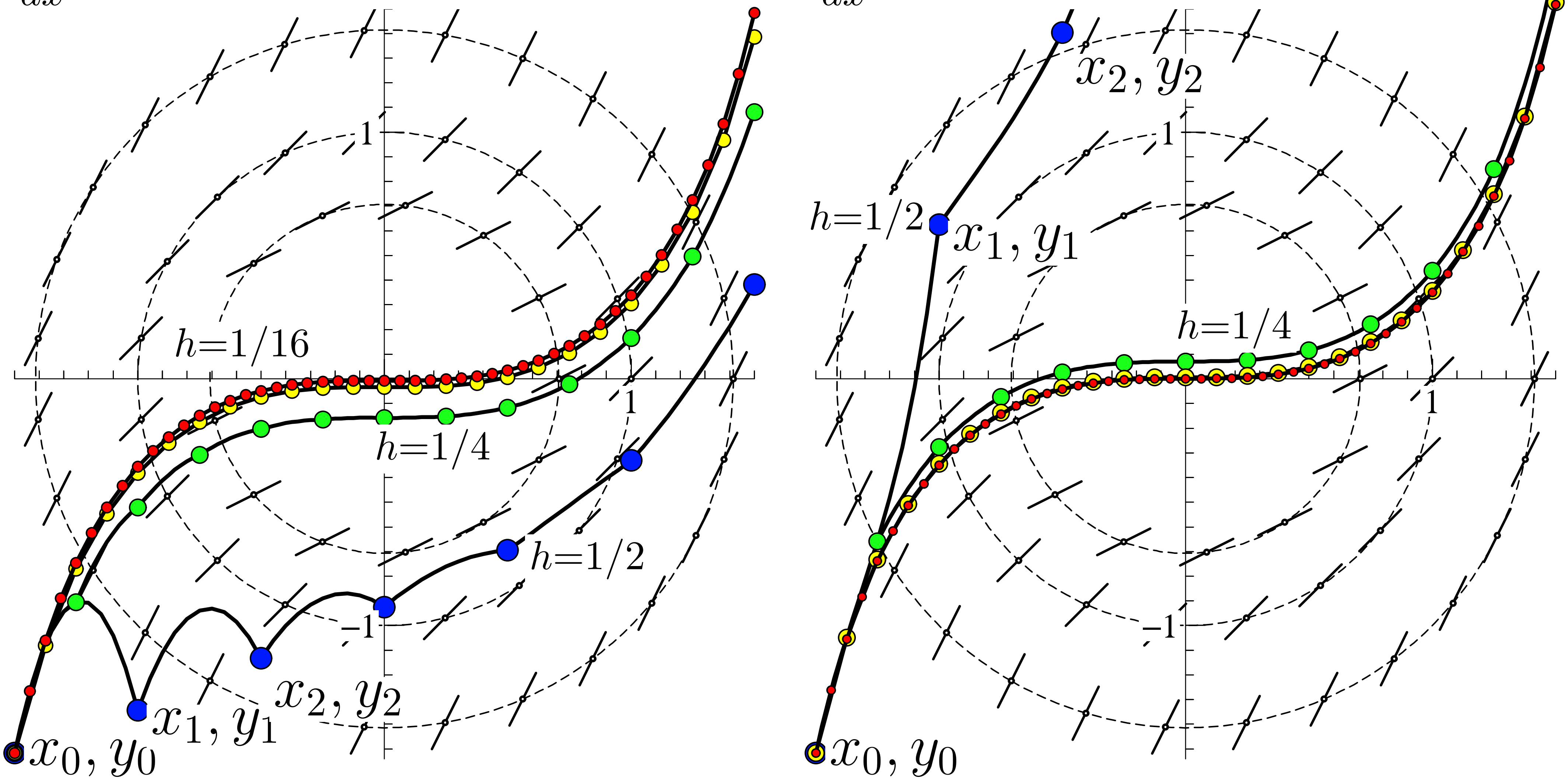


$$y_{n+1} = y_n + hy'_n$$



$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n$$

$\frac{dy}{dx} = x^2 + y^2$, Taylor 2 $\qquad$ $\frac{dy}{dx} = x^2 + y^2$, Taylor 3



$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n \qquad y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n$$

"Automatic differentiation". (A. Gibbons 1960, E. Fehlberg 1964, R.E. Moore 1966, W. Gautschi 1966,...)

Example: $y' = x^2 + y^2$

Set $y(x_0+h) = y_0 + hy_1 + h^2y_2 + \ldots,\;\; x^2 = x_0^2 + 2x_0h + h^2$,

$$\begin{aligned}
\text{develop:}\quad y' &= y_1 + 2y_2h + 3y_3h^2 + 4y_4h^3 + \ldots \\
&= x_0^2 + 2x_0h + h^2 \\
&\quad + y_0^2 + 2y_1y_0h + 2y_0y_2h^2 + 2y_0y_3h^3 \\
&\quad\quad y_1^2h^2 + 2y_1y_2h^3 + \ldots
\end{aligned}$$

$$\Rightarrow\;\; y_1 = x_0^2 + y_0^2,\;\; 2y_2 = 2x_0 + 2y_1y_0,\;\; 3y_3 = 1 + 2y_0y_2 + y_1^2, \ldots$$

"Automatic differentiation". (A. Gibbons 1960, E. Fehlberg 1964, R.E. Moore 1966, W. Gautschi 1966,. . .)

Example: $y' = x^2 + y^2$

Set $y(x_0+h) = y_0 + hy_1 + h^2y_2 + \ldots, \quad x^2 = x_0^2 + 2x_0h + h^2,$

$$\text{develop:} \quad \begin{array}{rlllll} y' = & y_1 & +2y_2h & +3y_3h^2 & +4y_4h^3 & + \ldots \\ = & x_0^2 & +2x_0h & +h^2 & & \\ & +y_0^2 & +2y_1y_0h & +2y_0y_2h^2 & +2y_0y_3h^3 & \\ & & & y_1^2h^2 & +2y_1y_2h^3 & + \ldots \end{array}$$

$$\Rightarrow \quad y_1 = x_0^2 + y_0^2, \;\; 2y_2 = 2x_0 + 2y_1y_0, \;\; 3y_3 = 1 + 2y_0y_2 + y_1^2, \ldots$$

$$\alpha = aa + bb,\; 2\beta = 2\alpha b + 2a,\; 3\gamma = 2\beta b + \alpha\alpha + 1,$$
$$4\delta = 2\gamma b + 2\alpha\beta,\; 5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta$$
$$6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma, \text{ etc.}$$

Surprise: Euler invented this too !! (E342), §663.

Second order equations (Euler E366, 1769, §1082)

$$\text{I. } \frac{2ddx}{dt^2} = \frac{X}{M}; \quad \text{II. } \frac{2ddy}{dt^2} = \frac{Y}{M}; \quad \text{III. } \frac{2ddz}{dt^2} = \frac{Z}{M}$$

$$\frac{ddx}{dt^2} = F(t,x) \quad \Rightarrow \quad \frac{dx}{dt} = v, \quad \frac{dv}{dt} = F(t,x) \quad \Rightarrow$$

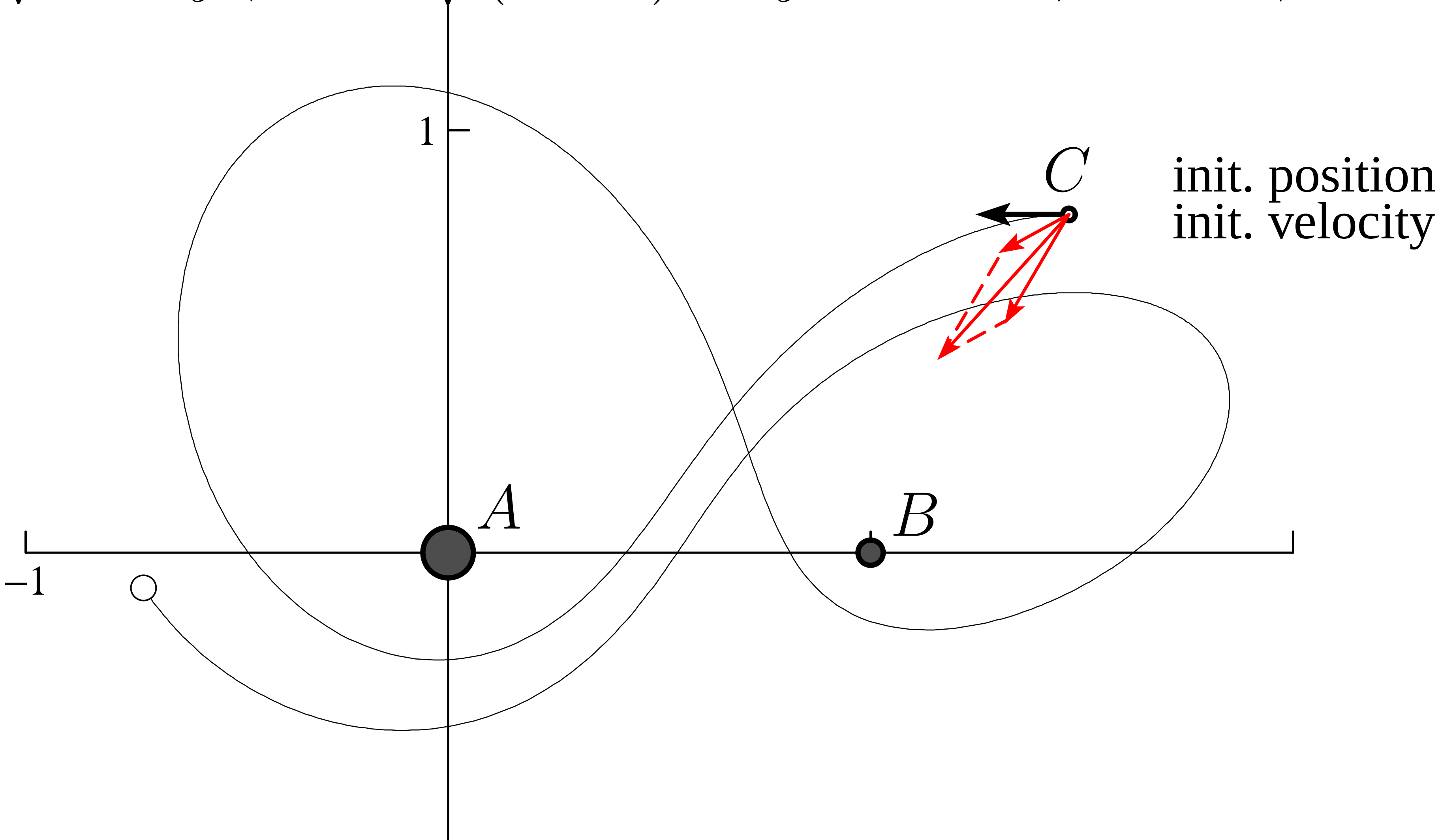
$$t_{n+1} = t_n + h, \quad x_{n+1} = x_n + hv_n, \quad v_{n+1} = v_n + hF(t_n, x_n),$$

$$x = a + \omega; \; y = b + c\omega; \; p = c + F\omega$$

Example: Particle attr. by two fixed centers (Euler E301, 1760)

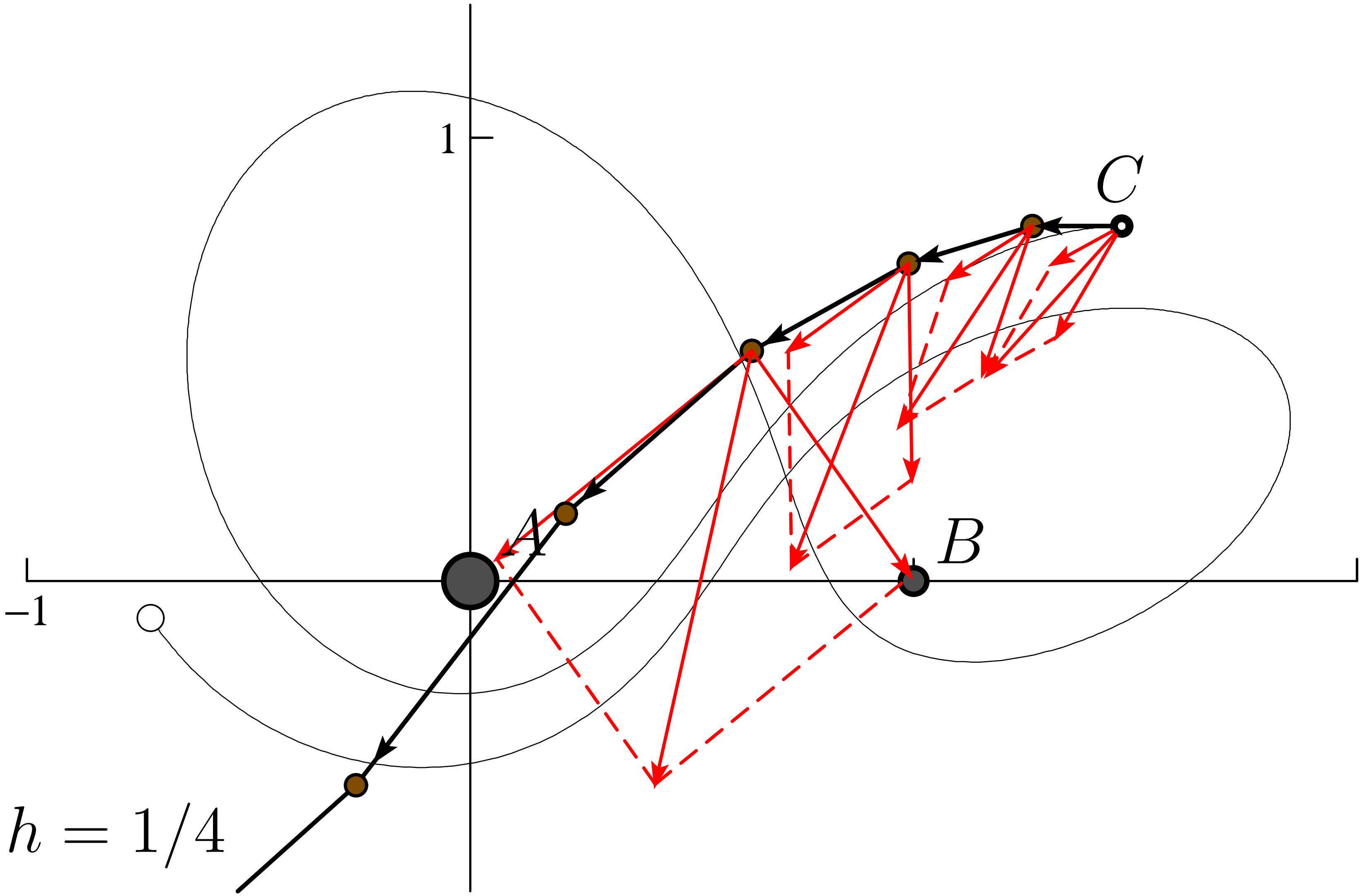
$$\frac{ddx}{dt^2} = -\frac{Ax}{v^3} - \frac{B(x-a)}{u^3}, \qquad \frac{ddy}{dt^2} = -\frac{Ay}{v^3} - \frac{By}{u^3}$$

$$v = \sqrt{x^2+y^2}, \quad u = \sqrt{(x-a)^2+y^2} \qquad A = 2, \quad B = 1, \quad a = 1$$



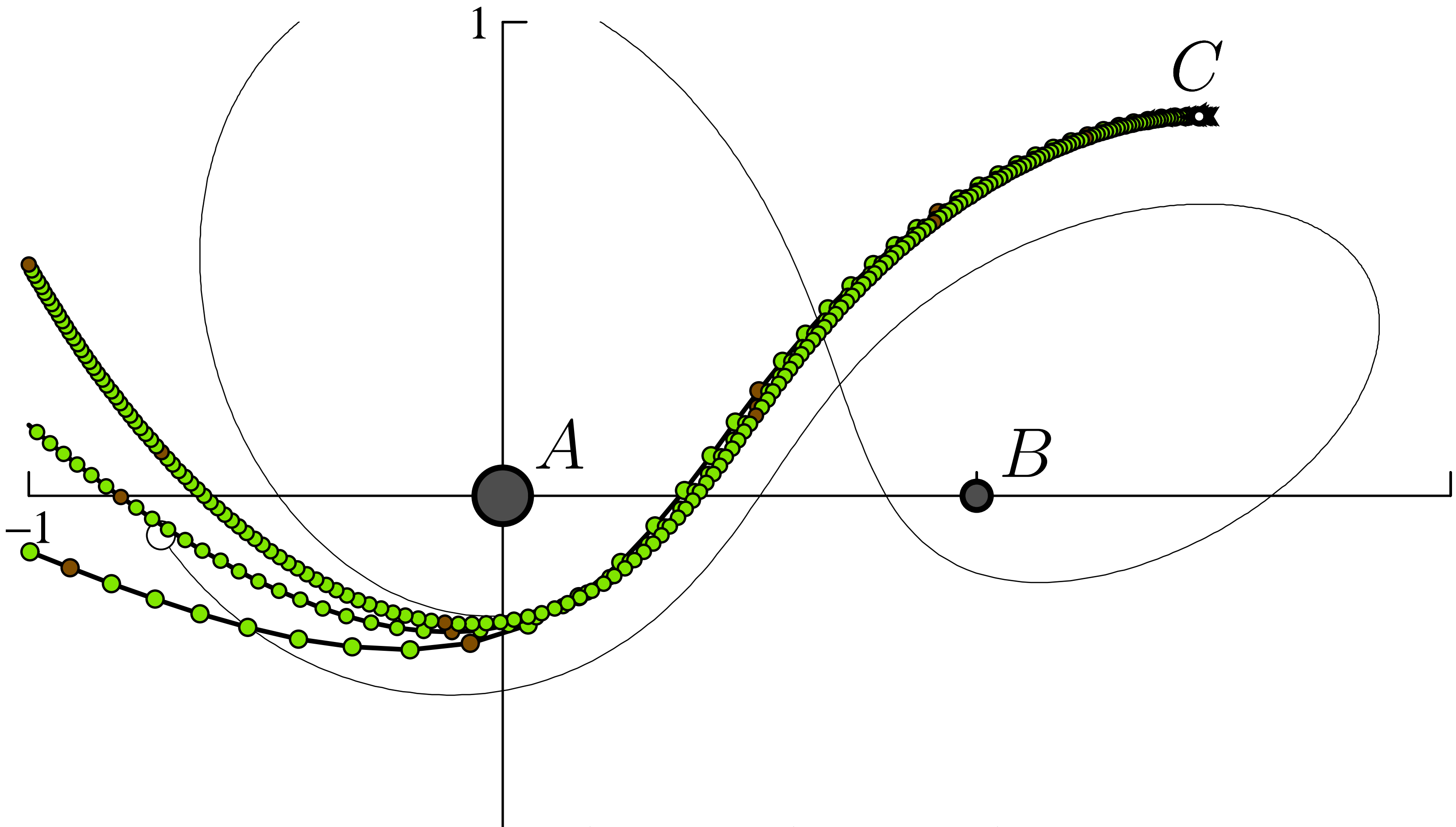
Euler's method:

$$x_{n+1} = x_n + hv_n, \quad v_{n+1} = v_n + hF(t_n, x_n),$$



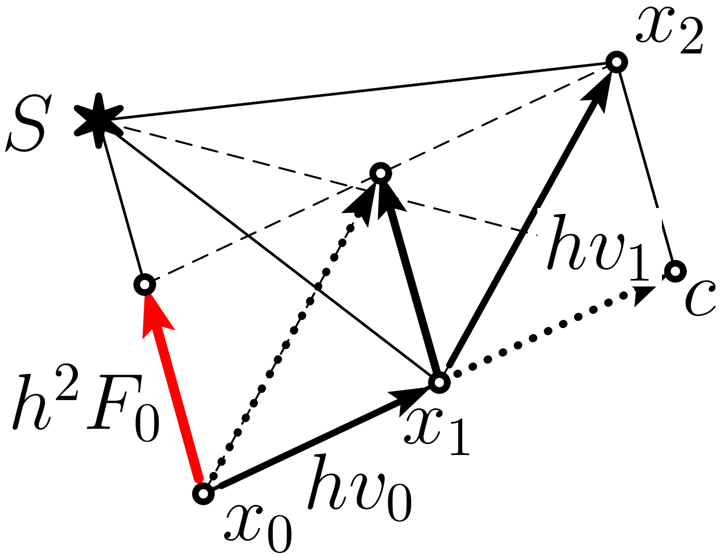
decrease step size:

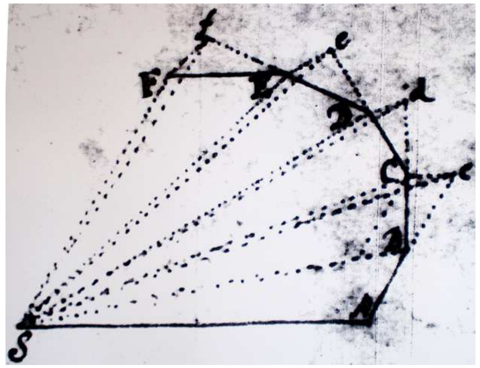
$$x_{n+1} = x_n + hv_n, \quad v_{n+1} = v_n + hF(t_n, x_n),$$

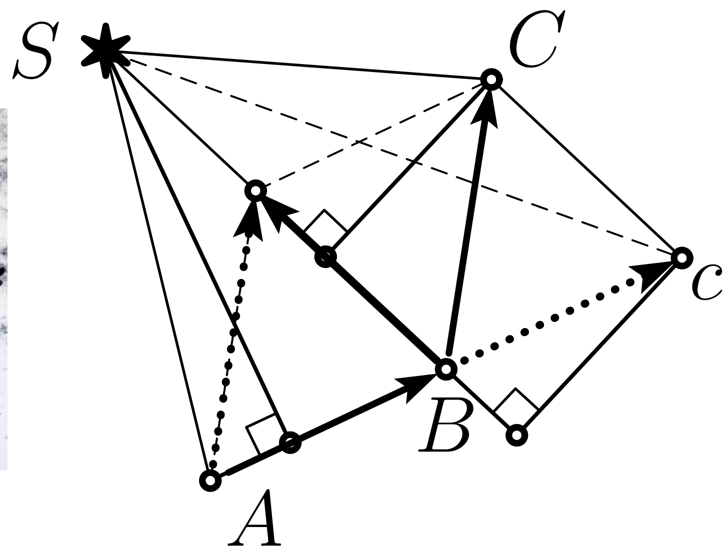


$$h = 1/32, \;\; 1/64, \;\; 1/128.$$

Learn from Newton (look at Kepler problem):







Newton

Euler

$$x_{n+1} = x_n + hv_n,$$
$$v_{n+1} = v_n + hF(t_n, x_n).$$

(areas not preserved)

"**symplectic** Euler" method

$$x_{n+1} = x_n + hv_n,$$
$$v_{n+1} = v_n + hF(t_{n+1}, x_{n+1}).$$

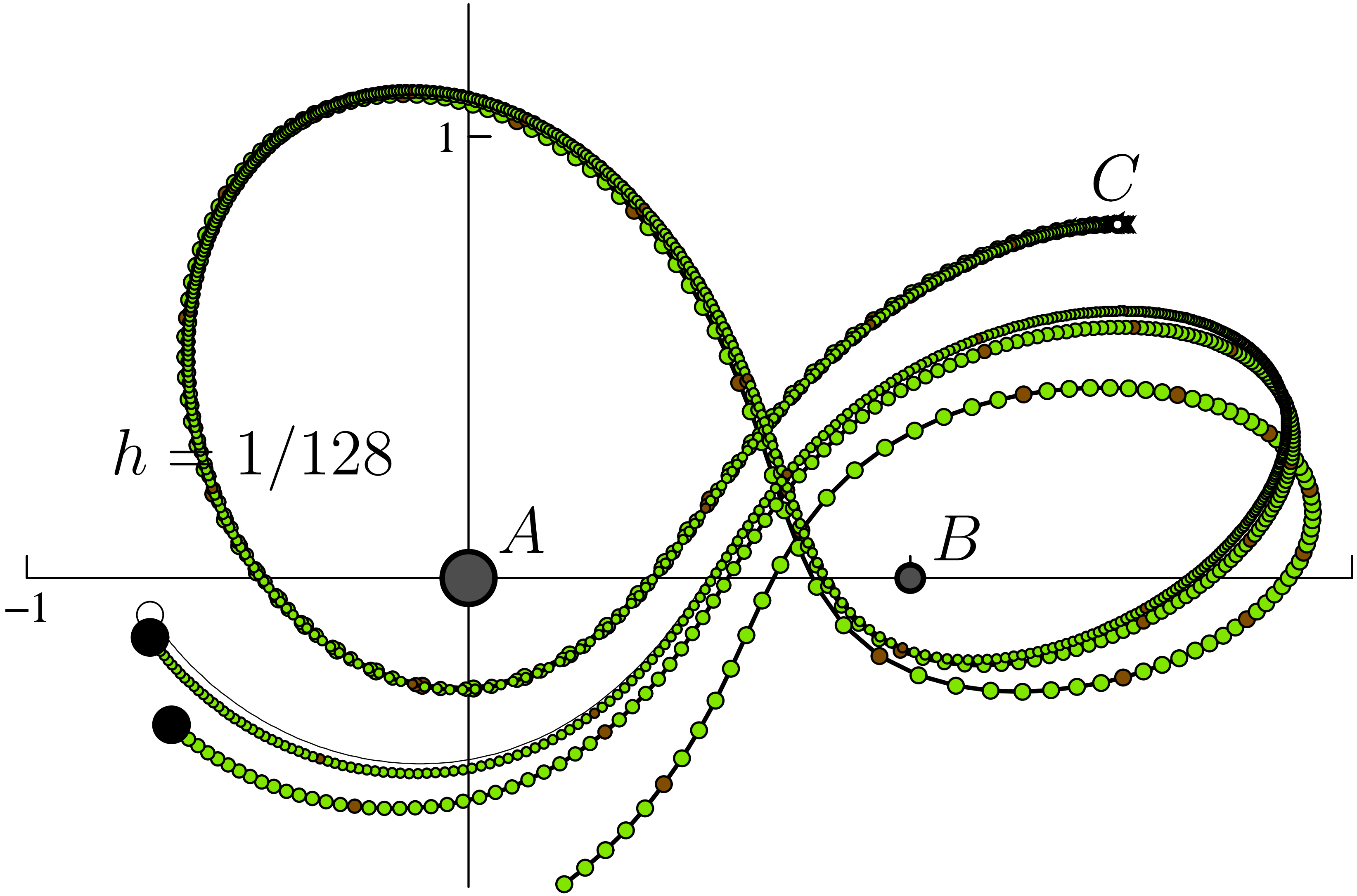
(areas preserved)

(is **geometric** integrator).

Funny: "Newton's equations" are due to Euler...

"Symplectic Euler method" is due to Newton.

Symplectic Euler method at two fixed centers:

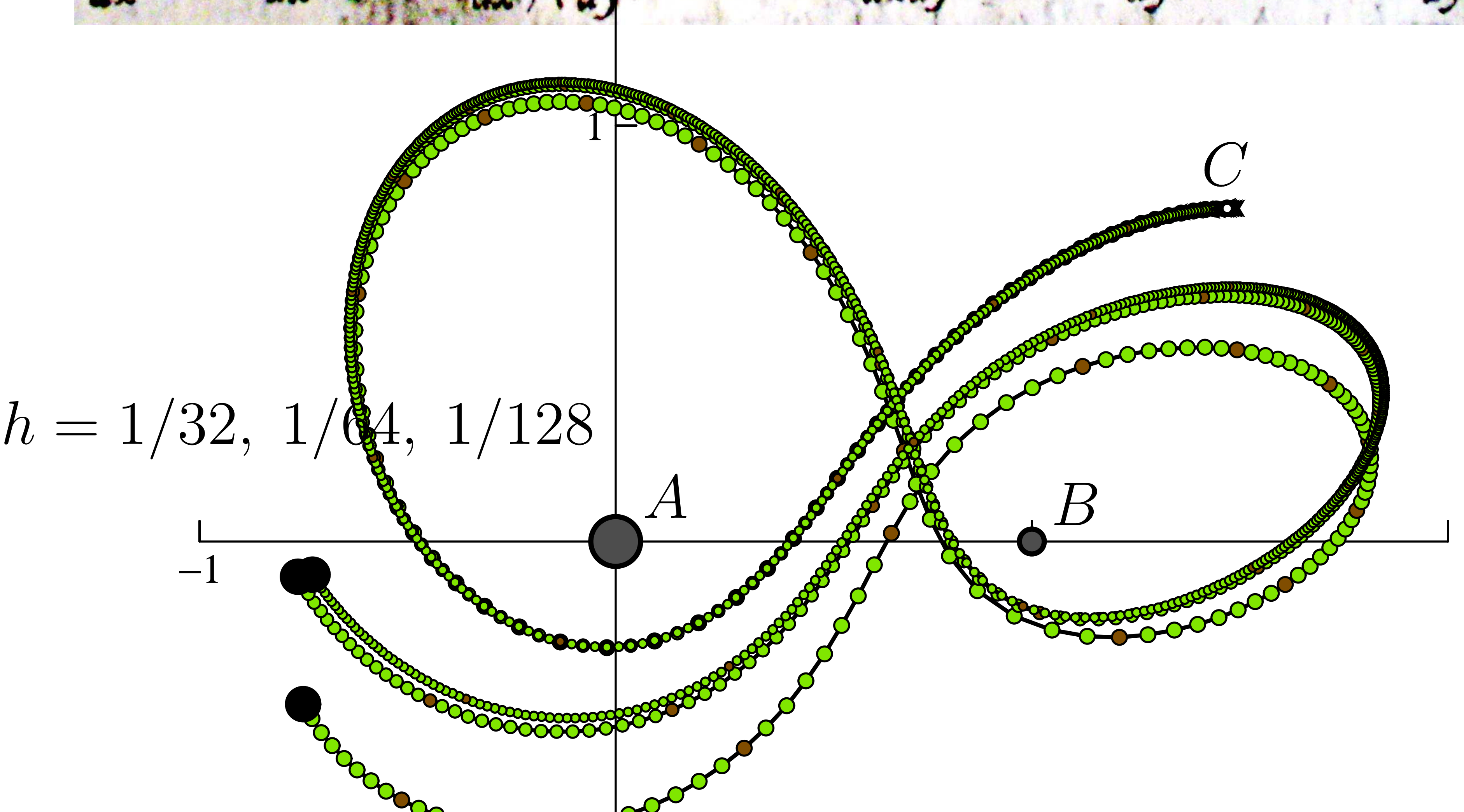


Third Order Method (Euler E342), ICI 1768, §656:

$$y = b + \frac{(x-a)\,db}{da} + \frac{(x-a)^2\,ddb}{1\cdot 2\,da^2} + \frac{(x-a)^3\,d^3b}{1\cdot 2\cdot 3\,da^3} + \text{etc.}$$

$$\frac{ddy}{dx^2} = \left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right)$$

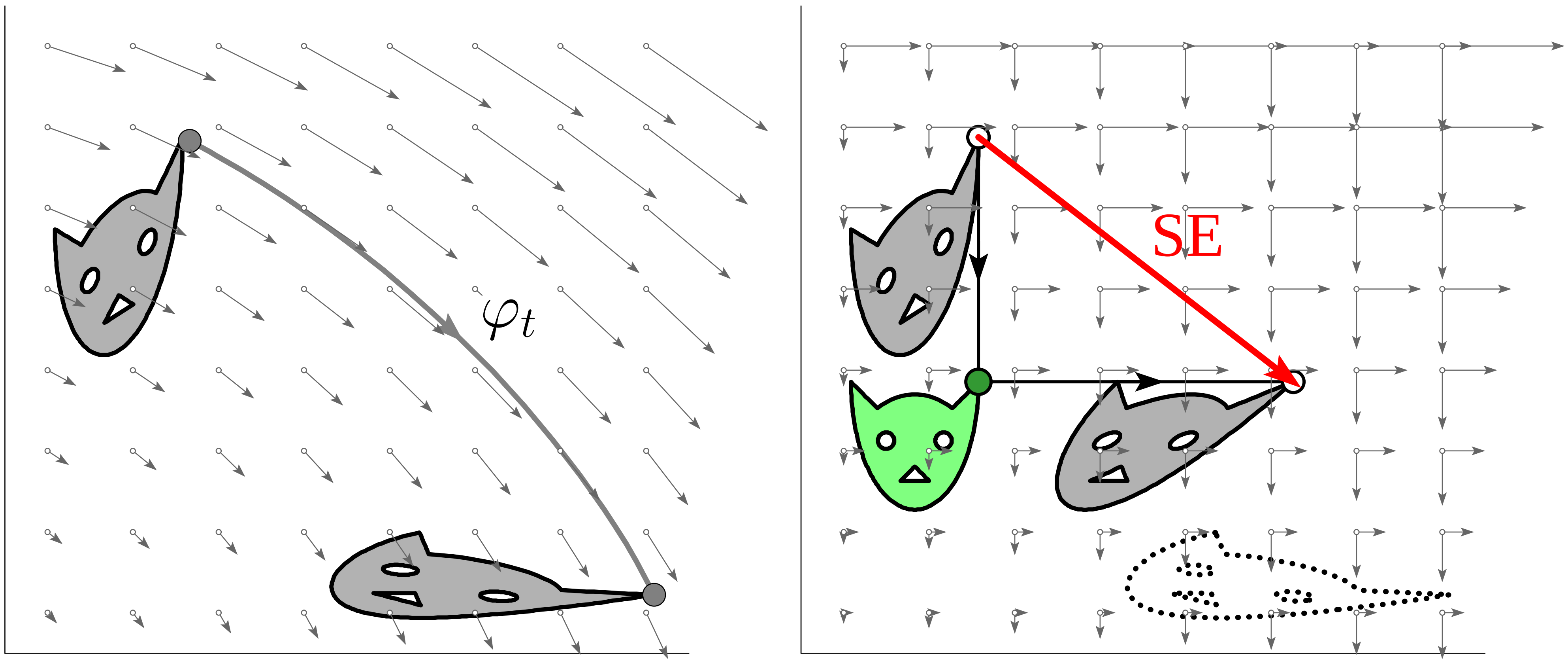
$$\frac{d^3y}{dx^3} = \left(\frac{ddV}{dx^2}\right) + \left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right) + 2V\left(\frac{ddV}{dx\,dy}\right) + V\left(\frac{dV}{dy}\right)^2 + VV\left(\frac{ddV}{dy^2}\right).$$



What is Symplecticity? (Poincaré 1899)

vary initial values ⇒ preserves (certain sums of) areas.

Example: Symplectic Euler method:



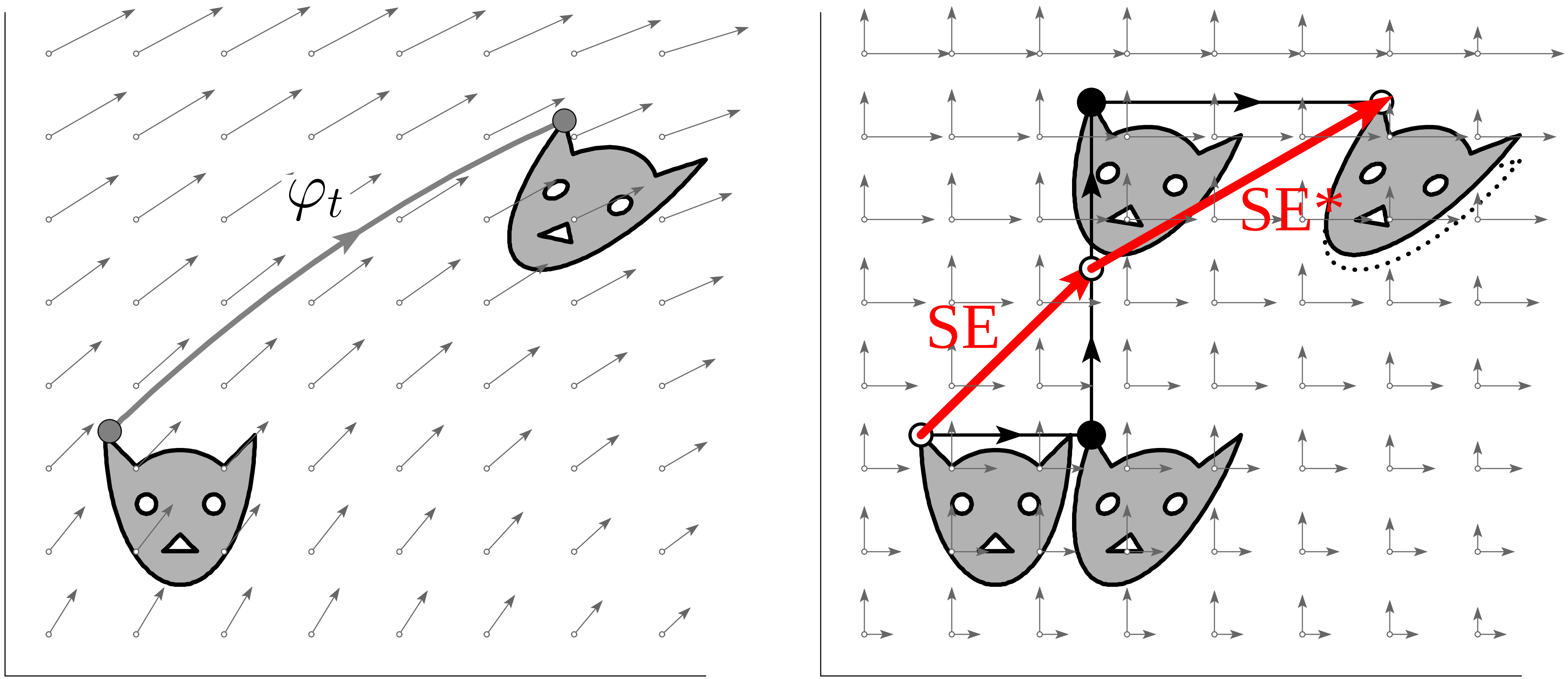
$$\dot{p} = -hH_q(p,q) \qquad \qquad p_{n+1} = p_n - hH_q(p_{n+1}, q_n)$$

$$\Rightarrow$$

$$\dot{q} = hH_p(p,q) \qquad \qquad q_{n+1} = q_n + hH_p(p_{n+1}, q_n)$$

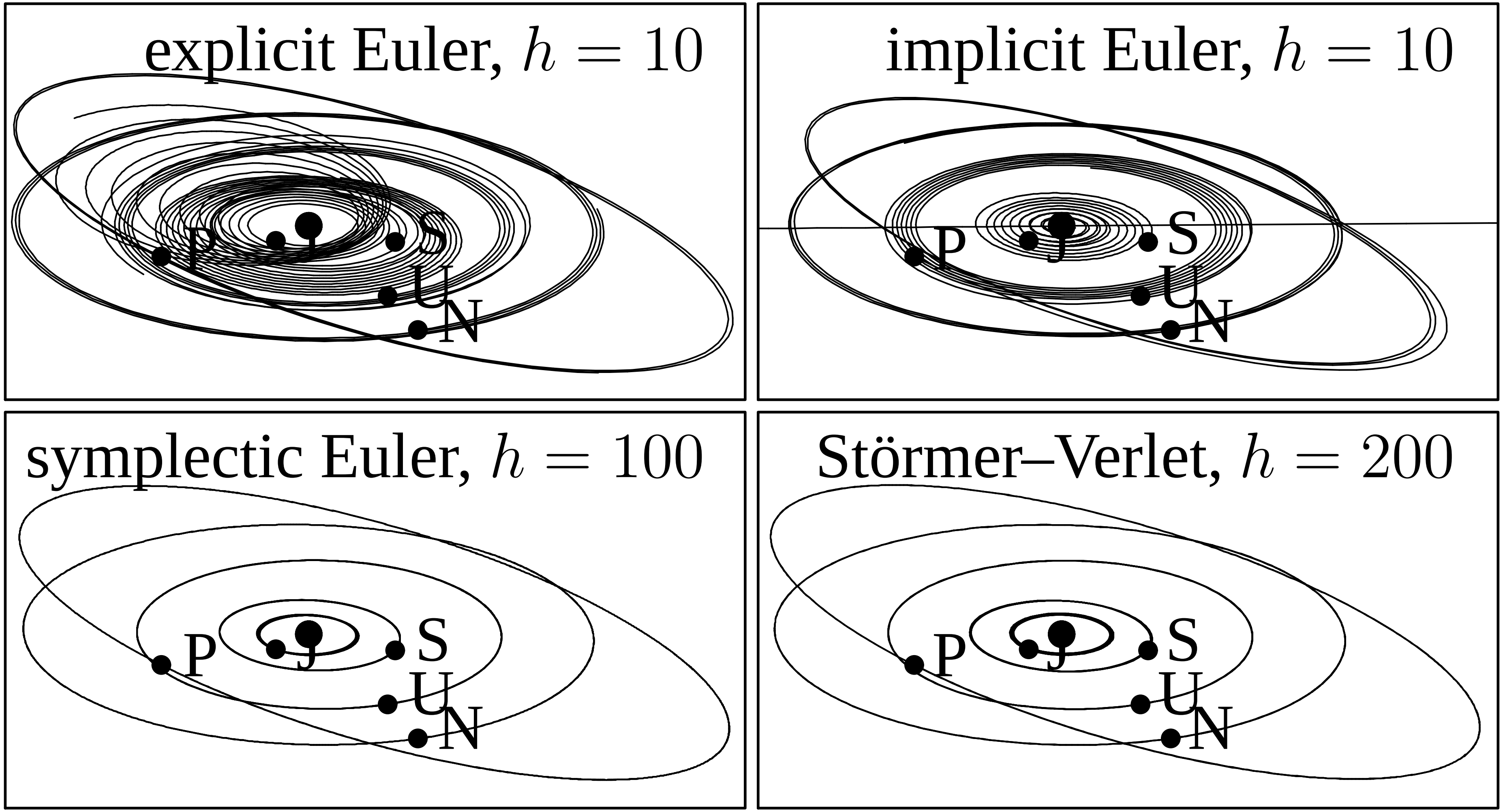
The Störmer-Verlet Method.

Störmer 1907 (Astronomy), Verlet 1967 (Molecular dynamics)

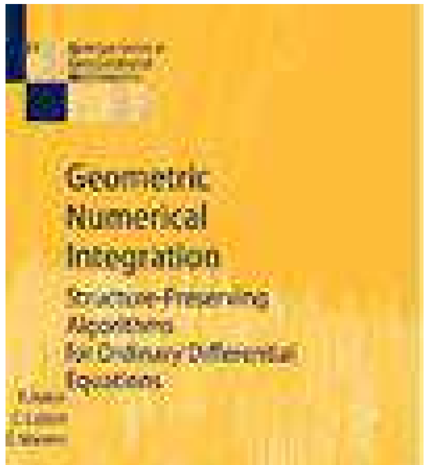


- Strang splitting :
- Composition of SE and SE* $\Rightarrow$ symplectic.

Example: Outer solar system.

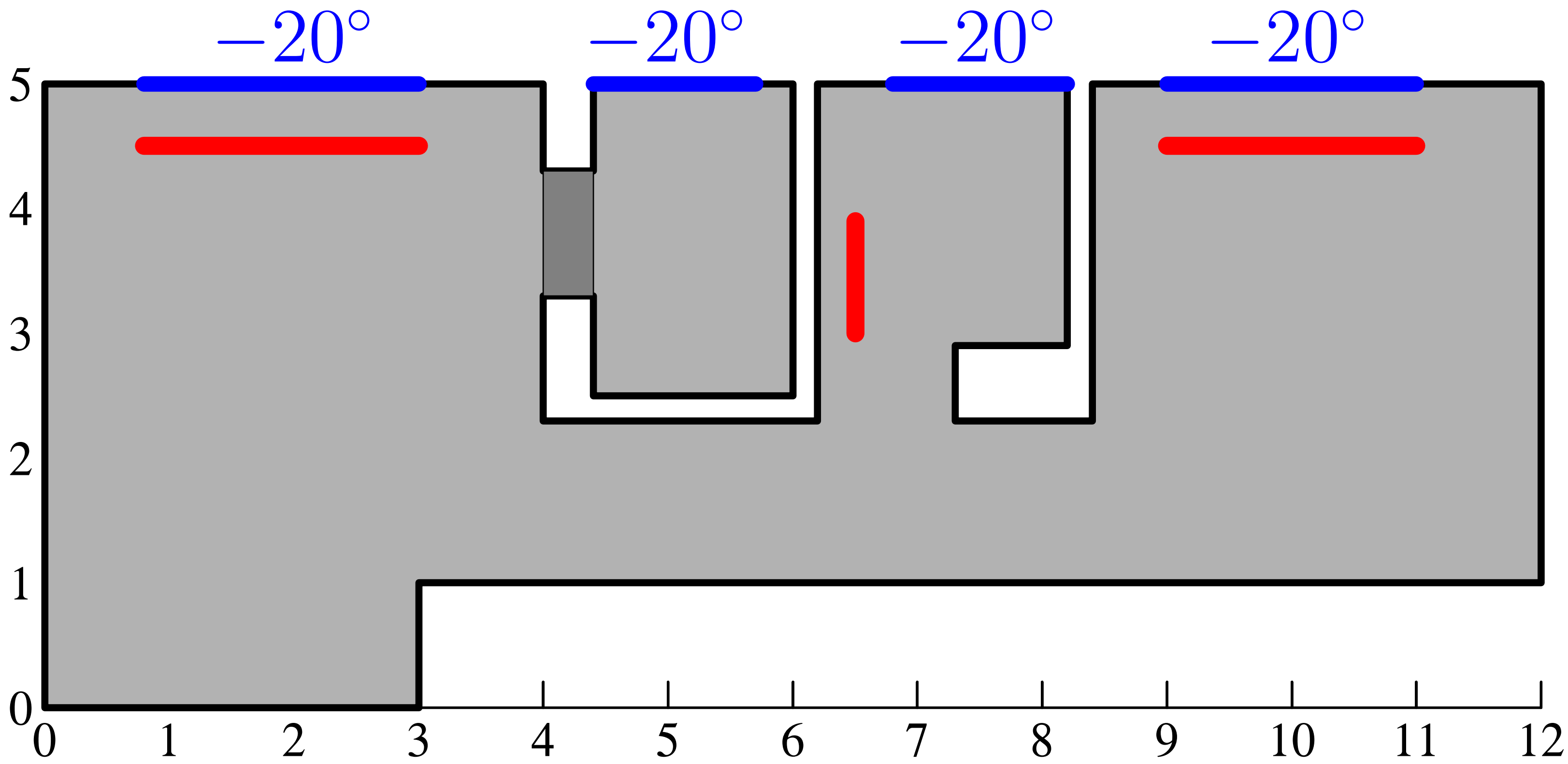


Pour plus de détails écoutez
conférences de Jacques Laskar ...
ou lisez ....



II. Ritz-Galerkin method:

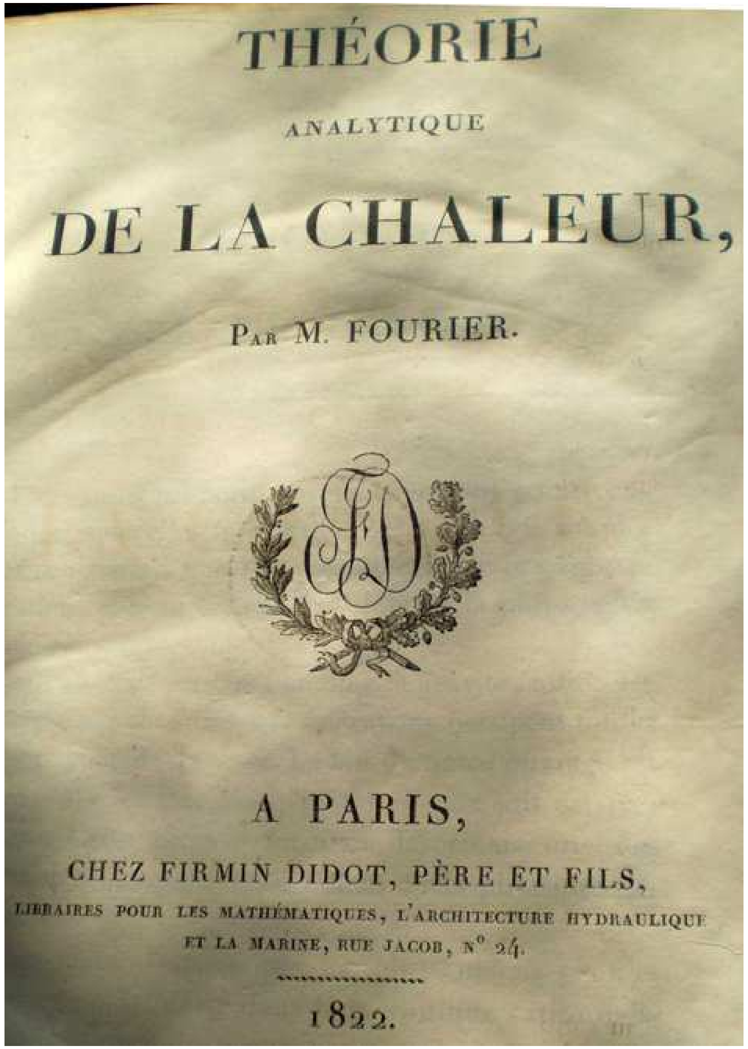
Problem. Appartment in Canadian winter at $-20°$



(Martin Gander)

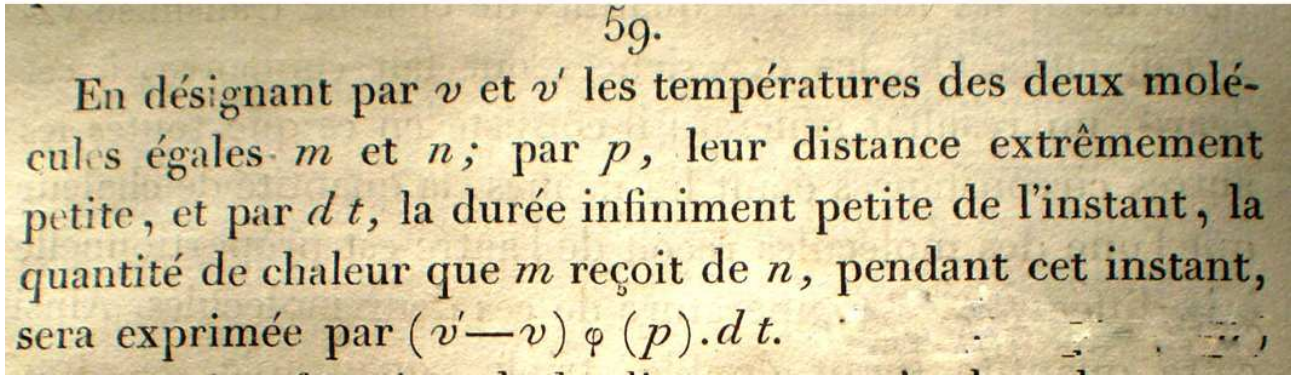
Find temperature everywhere !!

Solution: Fourier's Heat Equation (1807, 1822)



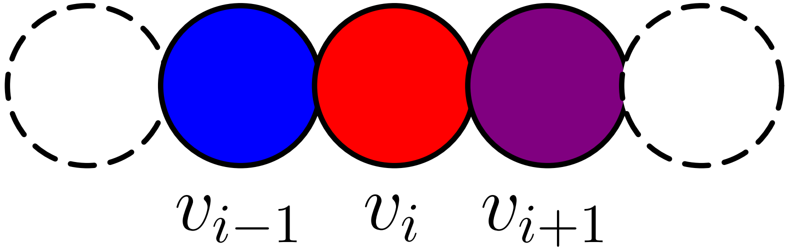
THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.

A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.
1822.

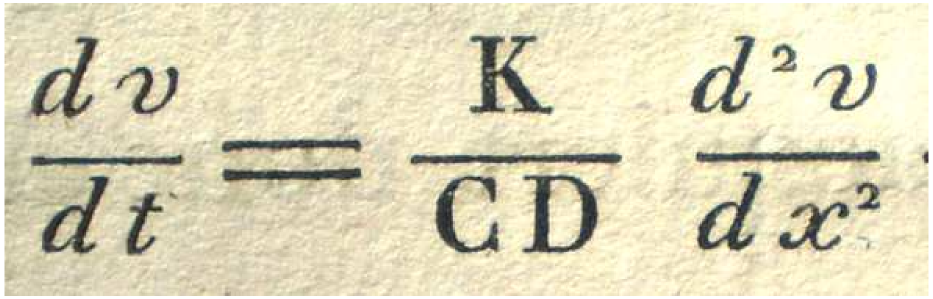


59.

En désignant par v et v' les températures des deux molécules égales m et n; par p, leur distance extrêmement petite, et par dt, la durée infiniment petite de l'instant, la quantité de chaleur que m reçoit de n, pendant cet instant, sera exprimée par $(v'-v)\,\varphi(p).dt$.



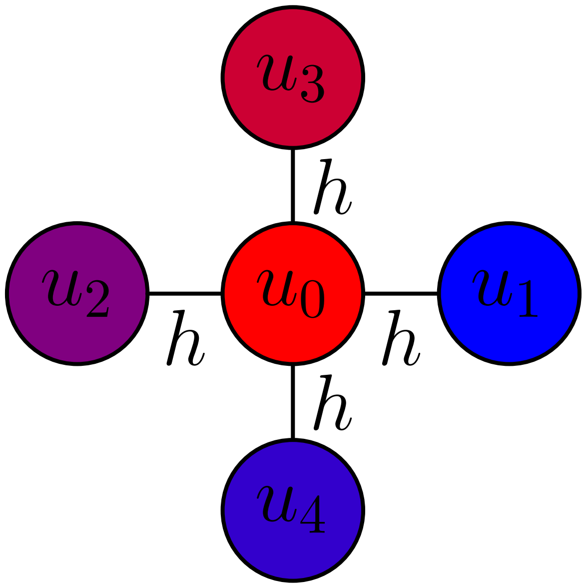
$$\frac{dv_i}{dt} = K \cdot ((v_{i+1} - v_i) - (v_i - v_{i-1}))$$



$$\frac{dv}{dt} = \frac{K}{CD}\frac{d^2v}{dx^2}.$$

Heat equation for 1-dimensional boby.

In two dimensions:



$$\frac{\partial u_0}{\partial t} = \frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{h^2} + f_0.$$

continuous problem $h \to 0$:

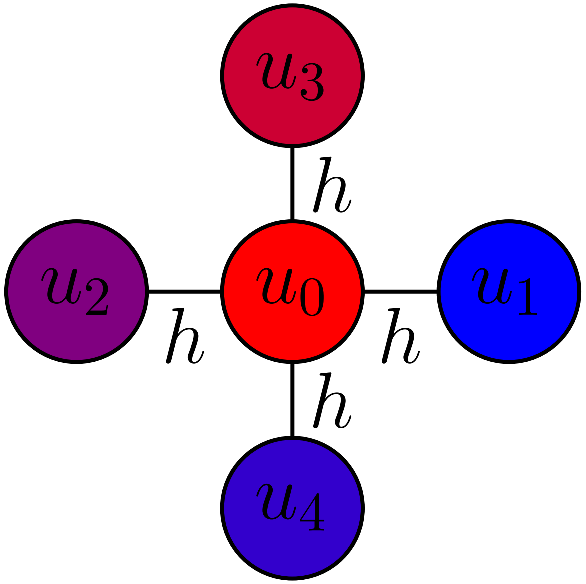
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y)$$

equilib.: $\boxed{-\Delta u = f}$

(f = source of heat).

How to solve ???

In two dimensions:



$$\frac{\partial u_0}{\partial t} = \frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{h^2} + f_0.$$

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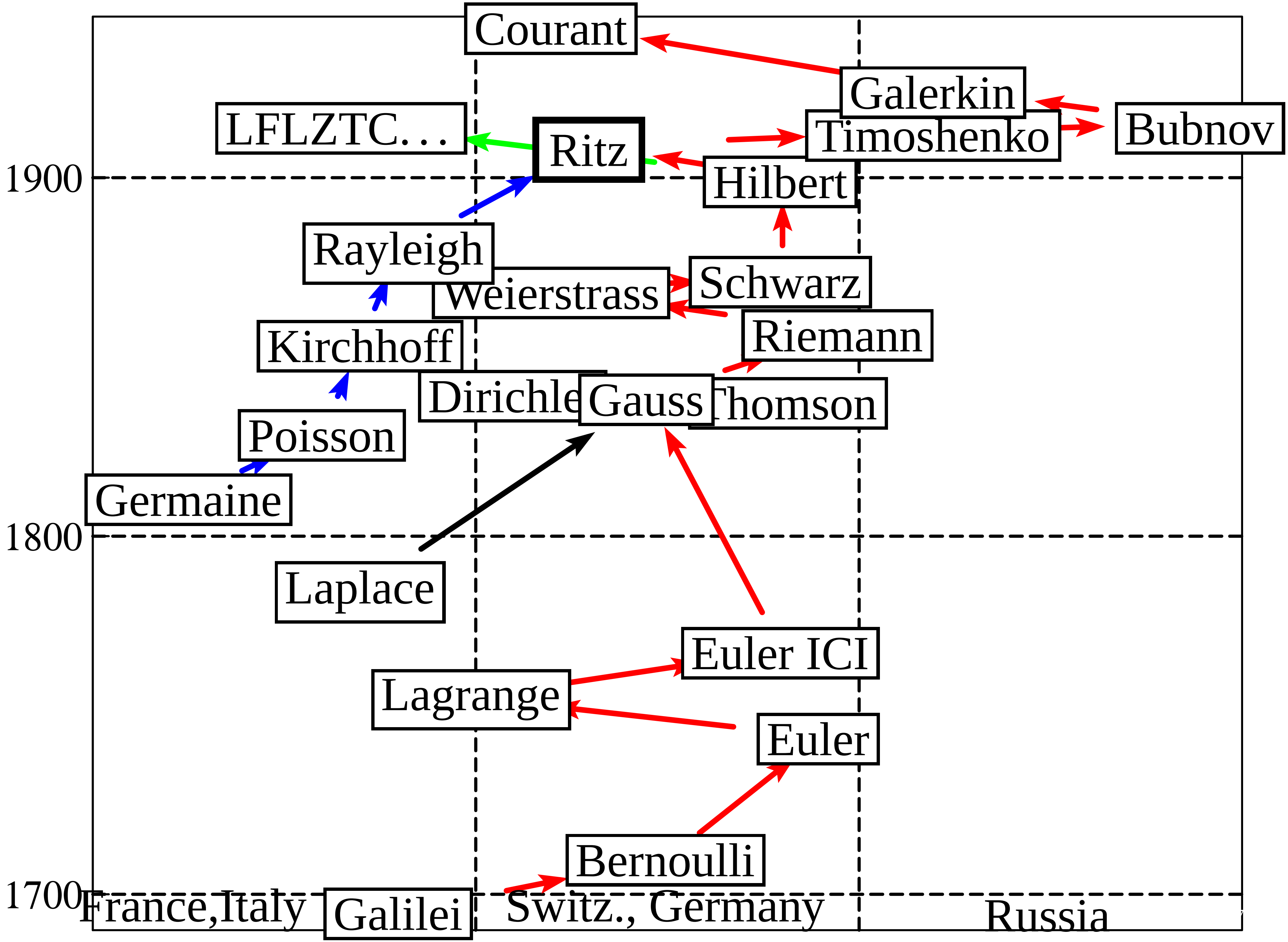
equilib.: $\boxed{-\Delta u = f}$

(f = source of heat).

How to solve ???

"Remember, Rome was not built in a day !!"

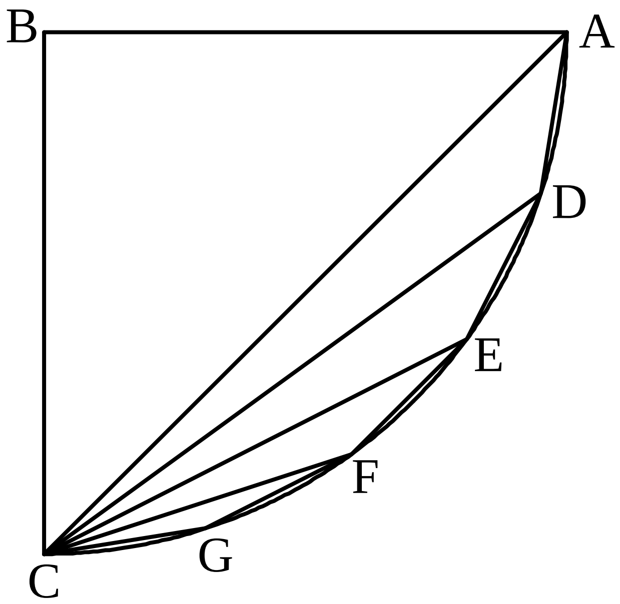
(Laurel & Hardy, *Dirty Work 1933)*



Courant
Galerkin
Bubnov
LFLZTC...
Ritz
Timoshenko
Hilbert
1900
Rayleigh
Weierstrass
Schwarz
Riemann
Kirchhoff
Dirichlet
Gauss
Thomson
Poisson
Germaine
1800
Laplace
Euler ICI
Lagrange
Euler
Bernoulli
1700
France,Italy
Galilei
Switz., Germany
Russia

Galilei. Discorsi 1638, Giornata terza, Teorema 22 e Scolie:

"Da quanto si è dimostrato sembra si possa ricavare che il movimento più veloce da estremo ad estremo non avviene lungo la linea più breve, cioè la retta, ma lungo un arco di cerchio."

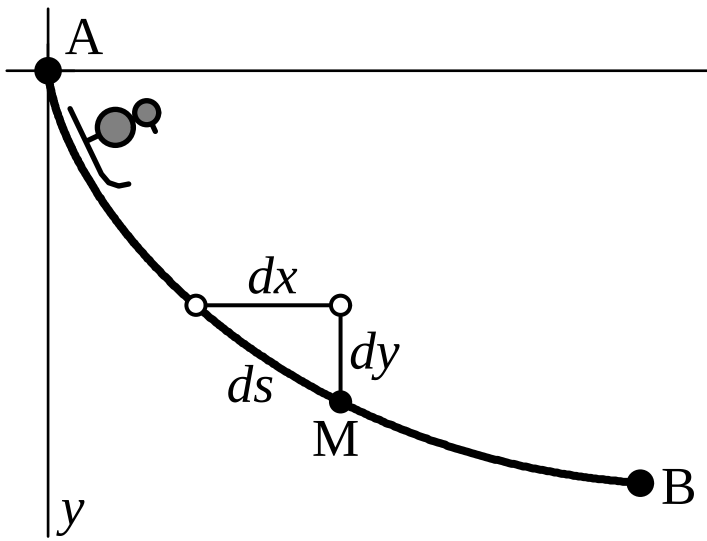


"... Mr. Leibnits remarque en Galilée deux fautes considerables: c'est que cet homme-là, qui étoit, sans contredit, le plus clairvoyant de son tems dans cette matiére, vouloit conjecturer que la courbe de la chainette étoit une Parabole, que celle de la plus vite descente étoit un Cercle.."
(Joh. Bernoulli, 1697)

Bernoulli's Brachystochrone.

PROBLEMA NOVUM Ad cujus solutionem Mathematici invitantur. Datis in plano verticali duobus punctis A & B, assignare Mobili M viam AMB, per quam gravitate sua descendens, & moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B.

(Joh. Bernoulli, Acta Erud., Jun. 1696)



$$\int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \text{min.}$$

$$\int \sqrt{\frac{1+p^2}{y}}\, dx = \text{min.} \quad (p = y').$$

"on fait tout avec les actions . . . en physique . . . tout !"

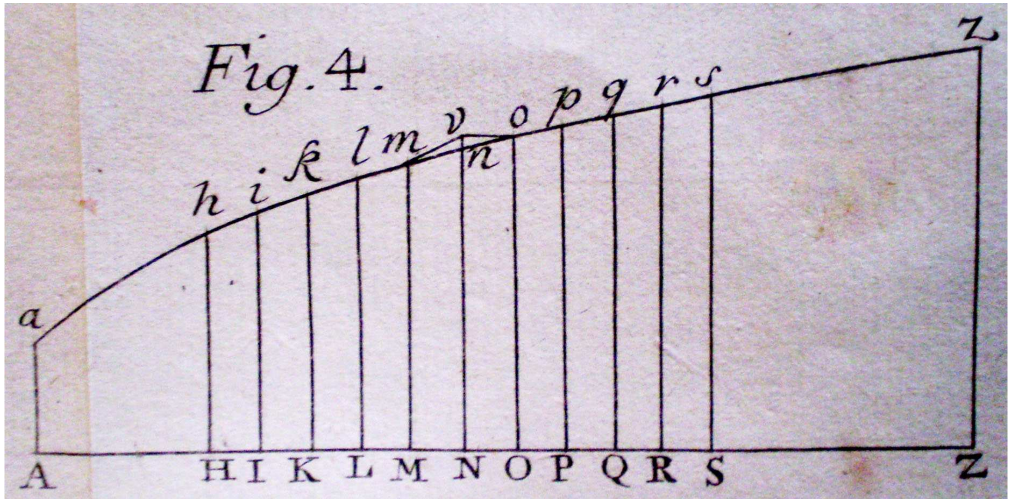
(K. Zuleta, giovedi, ore 11:32)

Euler (E65, 1744):

$$J = \int_a^b Z\,dx = \text{min! vel max!} \quad \text{where } Z = Z(x, y, p),\ \ p = \frac{dy}{dx}$$

Euler's Solution.

1. Approximate curve by polygon



2. Approx. Integral by -sum ("Riemann")

$$Z\,dx + Z'\,dx + Z''\,dx + Z'''\,dx + \text{\&c.}$$

and diff. w.r. to ν;

3. Set derivative

$$(P + N'dx - P')$$

to zero;

4. inverse Euler method $\Rightarrow$

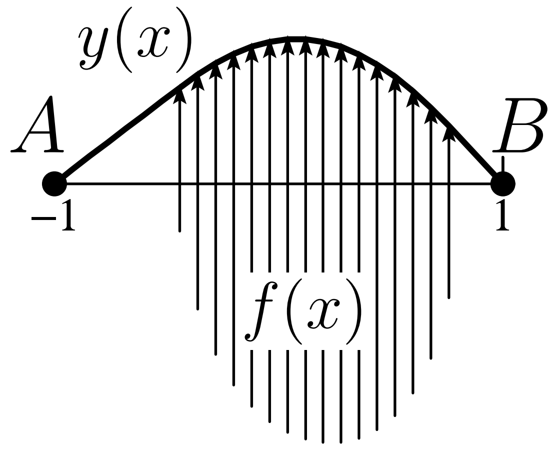
$$N - \frac{dP}{dx} = 0$$

$\left(N = \frac{\partial Z}{\partial y},\ P = \frac{\partial Z}{\partial p}\right).$

Example 1.

$$J = \int_a^b \left(\frac{p^2}{2} - f \cdot y \right) dx \longrightarrow \min$$

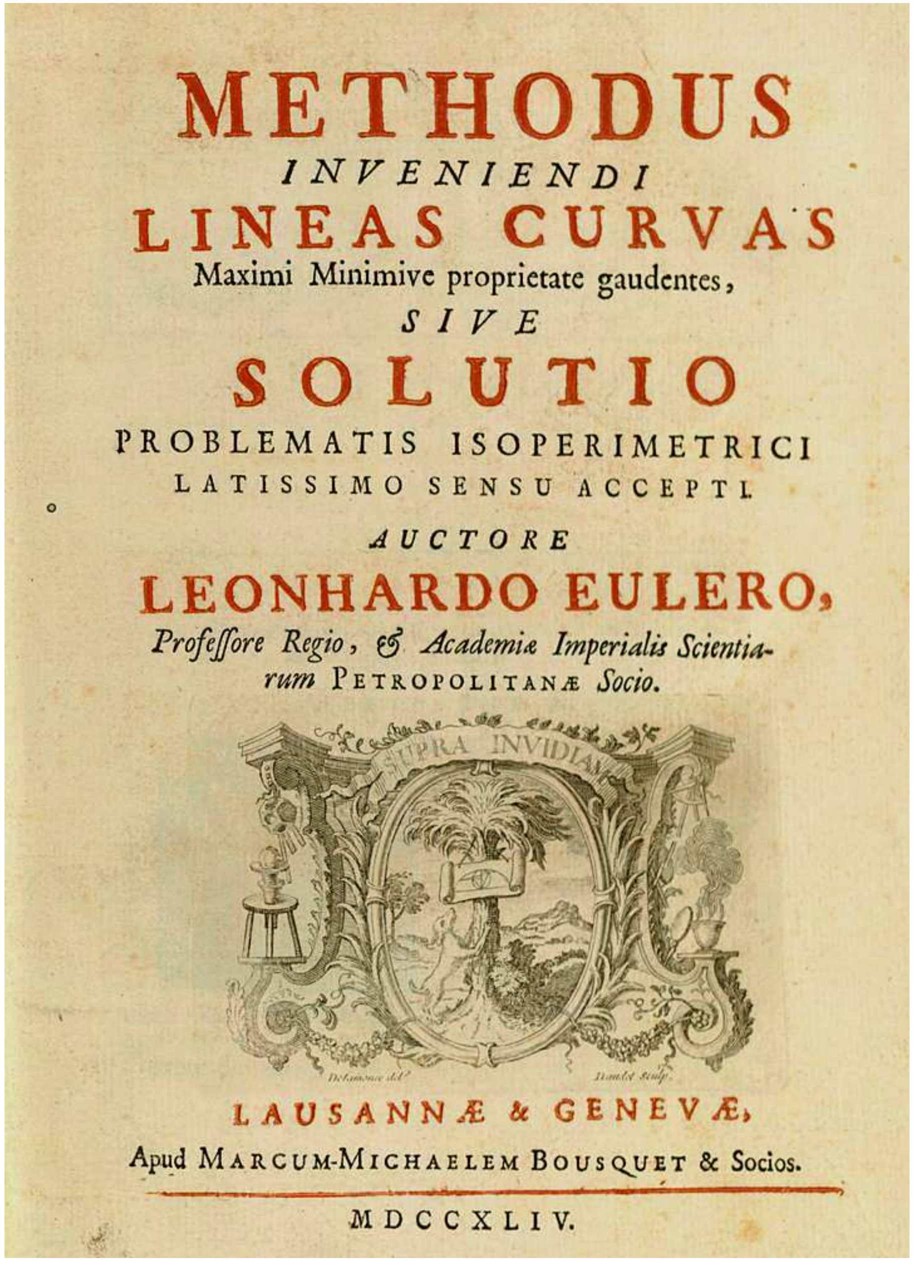
$$-\frac{d^2 y}{dx^2} = f(x)$$ "Newton's eqs."



Example 2. (the Brachystochrone ; Euler E65, Caput II, §34).

$$\frac{\sqrt{1+p^2}}{\sqrt{2gy}} - \frac{p^2}{\sqrt{2gy}\,\sqrt{1+p^2}} = C \quad \text{or} \quad 1 = \sqrt{1+p^2}\sqrt{2gy} \cdot C\ .$$

same integral as in M. Kunz' talk on expansion of the Univers
$\Rightarrow$ Cycloide.



METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*

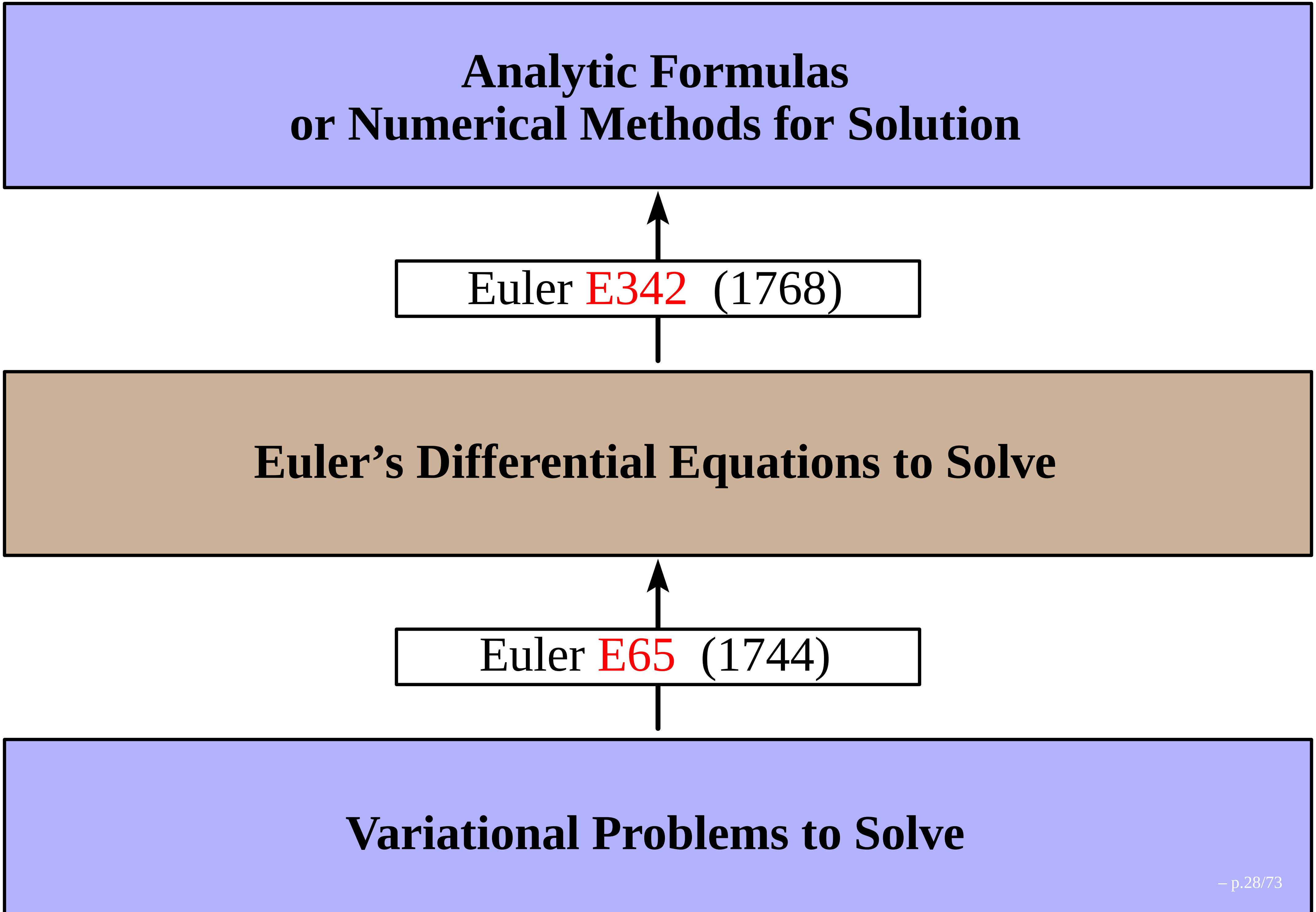
LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

Variational Calc. 1744 E65

INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN PRIMVM
IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.
AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LONDIN.
PETROPOLI
Impensis Academiae Imperialis Scientiarum
1768.

Inst. Calc. Integralis 1768 E342

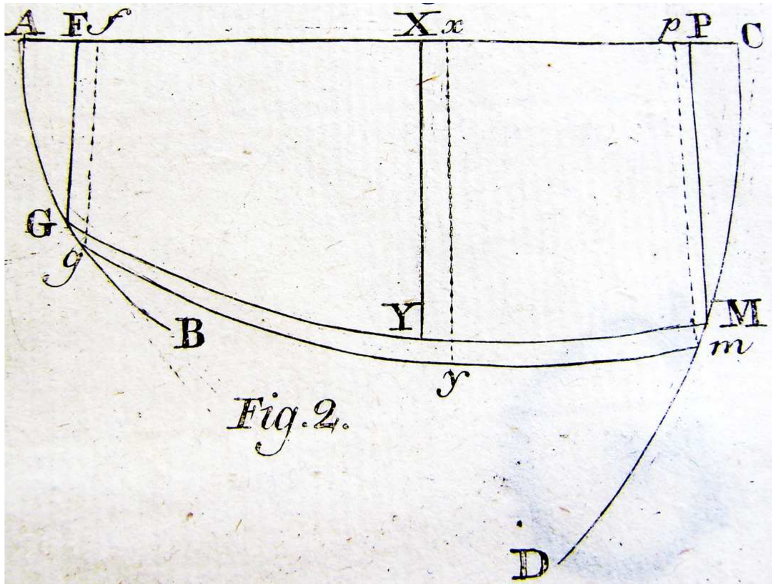
Euler’s “World”:



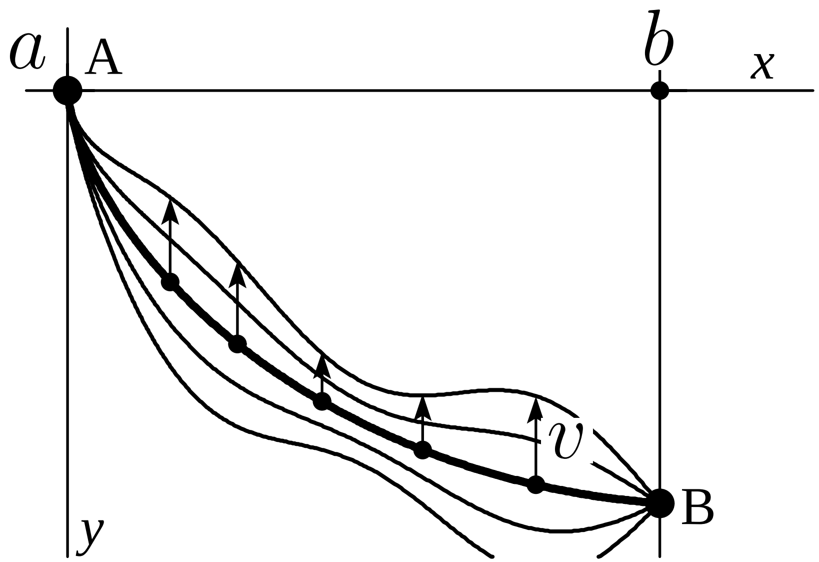
Joseph Louis de Lagrange 1755:

Ludovico de la Grange Tournier (19 years) writes 12 aug. 1755 to *Vir amplissime atque celeberrime L. Euler*. From whom *Vir praestantissime atque excellentissime Lagrange* receives a kind and enthusiastic answer (6 september 1755).

Lagrange simplifies the derivation of Euler's equations by the Variational Calculus notation δy, which varies all values of y simultaneously:



(Picture from Euler, ICI vol. 3, 1770, Appendix)



Euler E420 (1772)

Euler E420 (1772) “Methodus nova et facilis . . .”

$$J(\varepsilon) = \int_a^b Z(x, y + \varepsilon v, p + \varepsilon v')\, dx = \text{min!}\big|_{\varepsilon=0}$$ Variat. Probl.

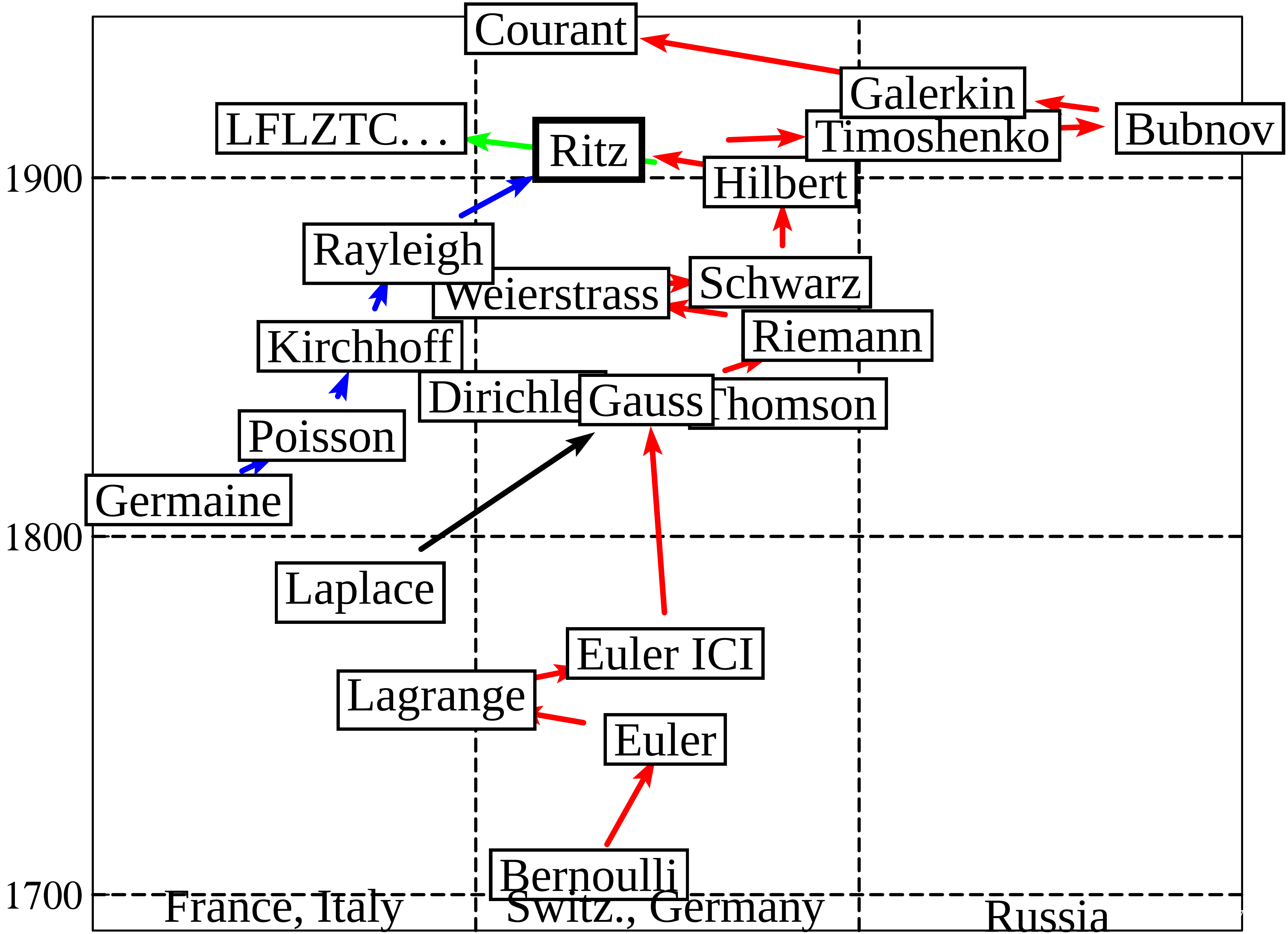
Differentiate: $$\frac{\partial J(\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \int_a^b (N \cdot v + P \cdot v')\, dx = 0.$$

Integration by parts:

$$\int_a^b \left(N - \frac{d}{dx}P\right) \cdot v \cdot dx = 0$$ “weak solution”

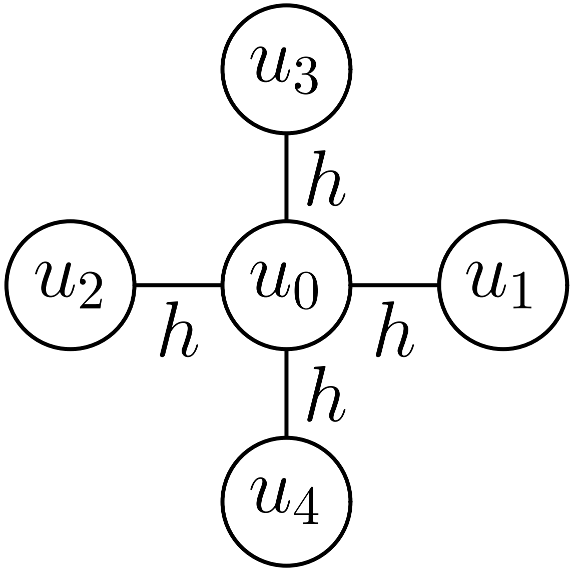
if $v(a) = v(b) = 0$. Hence, since v is arbitrary, $\Rightarrow$

$$N - \frac{d}{dx}P = 0$$ “strong solution”.


Courant
Galerkin
Bubnov
LFLZTC...
Ritz
Timoshenko
Hilbert
1900
Rayleigh
Weierstrass
Schwarz
Riemann
Kirchhoff
Dirichlet
Gauss
Thomson
Poisson
Germaine
1800
Laplace
Euler ICI
Lagrange
Euler
Bernoulli
1700
France, Italy
Switz., Germany
Russia

"Dirichlet's Principle" (**C.F. Gauss,** Werke 5, p. 195, 1839; **W. Thomson,** Liouville J. 12 (1847) p. 496).

$$\iint_\Omega \left(\tfrac{1}{2}\left(\left(\tfrac{\partial u}{\partial x}\right)^2 + \left(\tfrac{\partial u}{\partial y}\right)^2\right) - f\cdot u\right) dx\, dy = \text{min!} \quad \Rightarrow \quad -\Delta u = f.$$



Proof in Eulerian style: Derive

$$\frac{1}{2}\left(\ldots+\left(\frac{u_1-u_0}{h}\right)^2+\left(\frac{u_0-u_2}{h}\right)^2+\left(\frac{u_3-u_0}{h}\right)^2+\left(\frac{u_0-u_4}{h}\right)^2+\ldots\right)-f_0\cdot u_0.$$

with respect to u_0 and set derivative $= 0$:

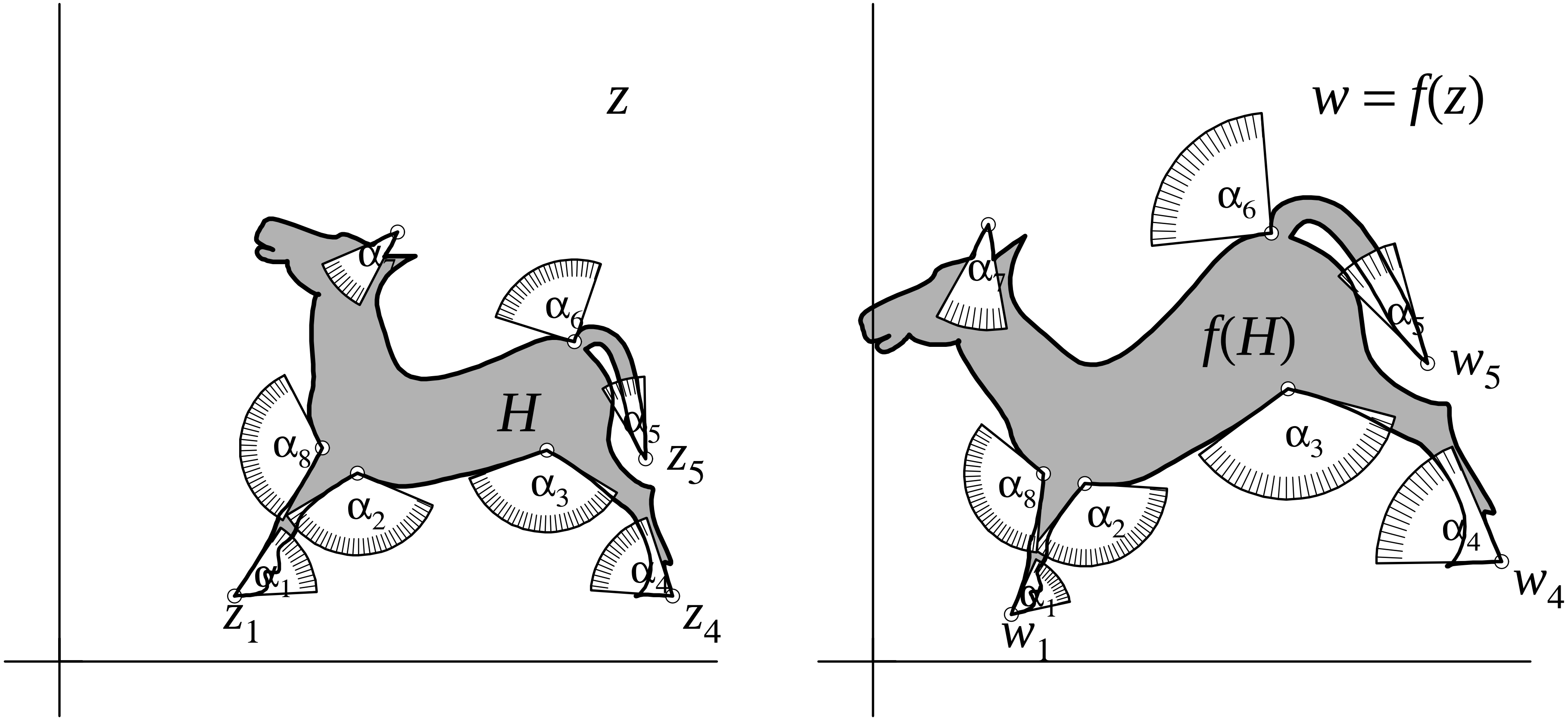
$$\frac{-u_1-u_2-u_3-u_4+4u_0}{h^2} - f_0 = 0. \quad \text{QED.}$$

B. Riemann (Thesis 1851) $f(z) = u(x,y) + iv(x,y)$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad f' = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

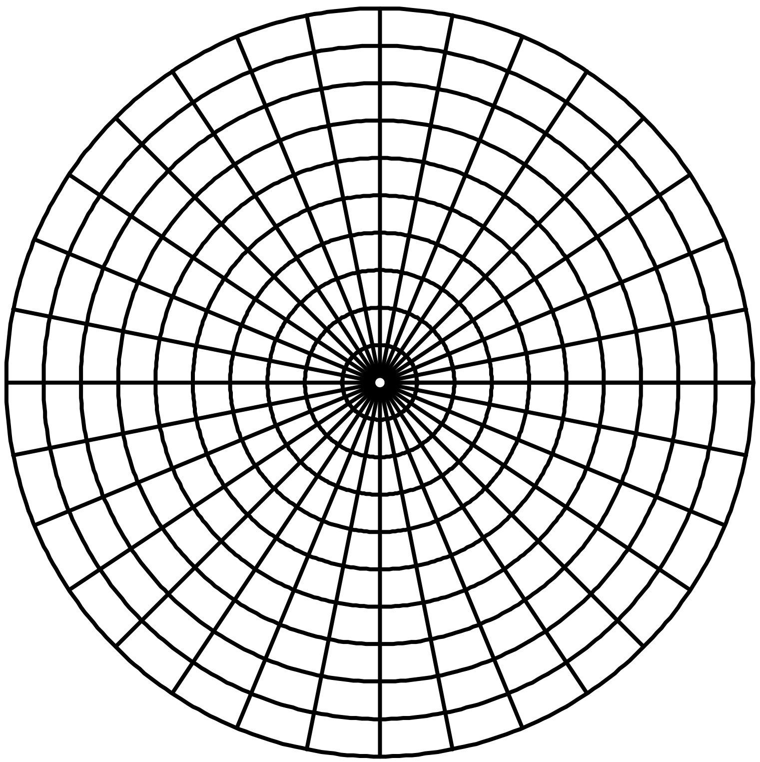
$\Rightarrow$ complex mapping is conformal (angle preserving).

“ ... und ihre entsprechenden kleinsten Theile ähnlich sind;” (Thesis §21)

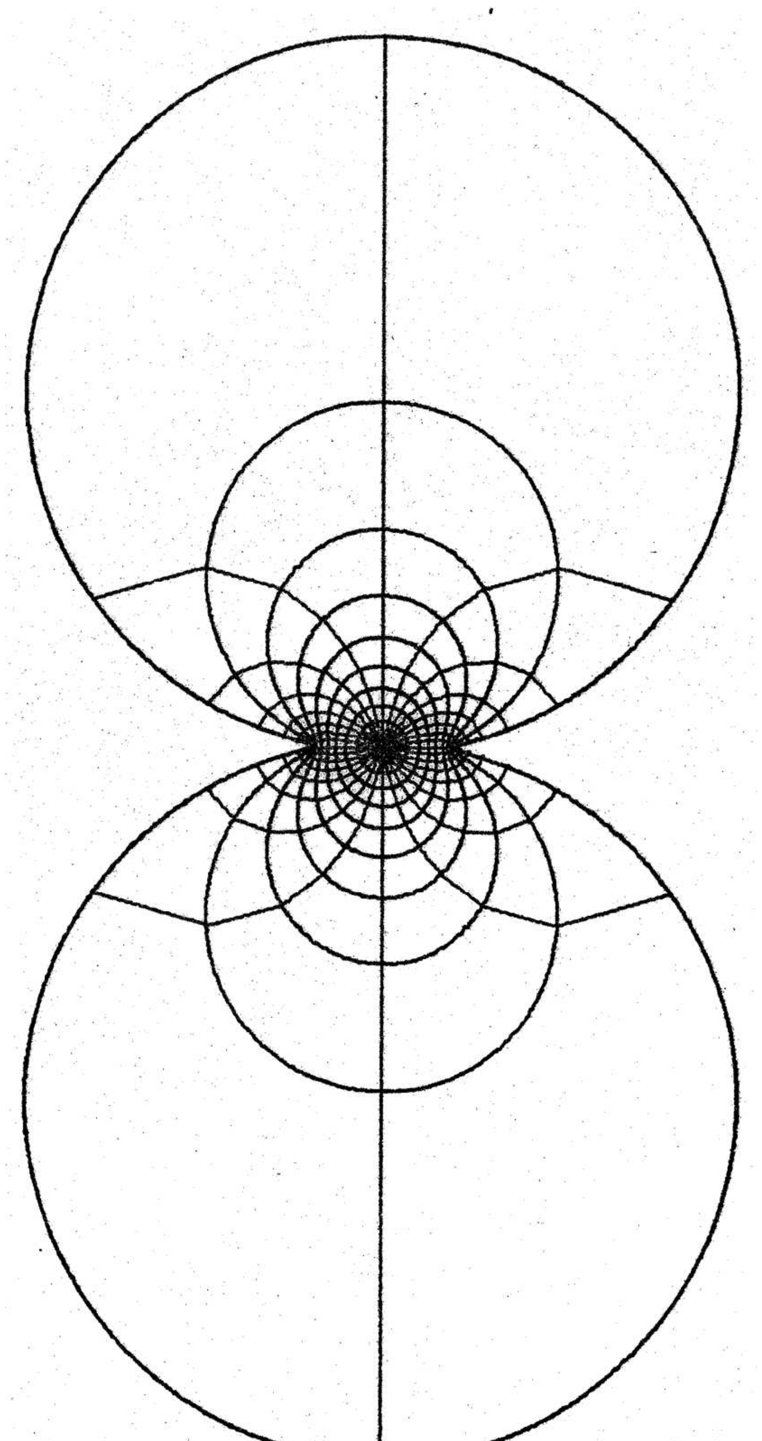


Riemann mapping theorem: (Thesis §21)

"Zwei gegebene einfach zusammenhängende Flächen können stets so aufeinander bezogen werden, dass jedem Punkte der einen Ein mit ihm stetig fortrückender Punkt entspricht...;"



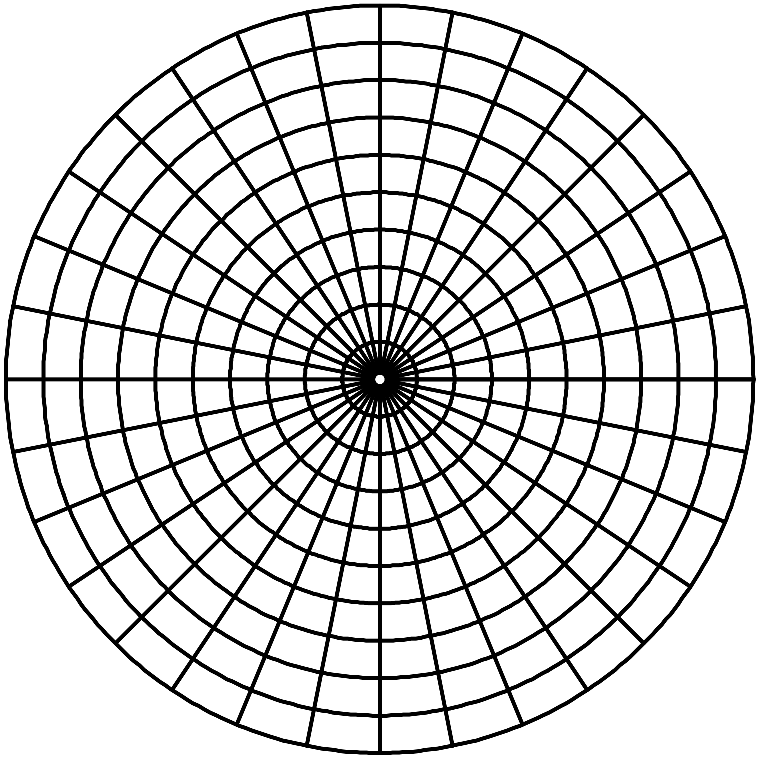
$\Leftrightarrow$



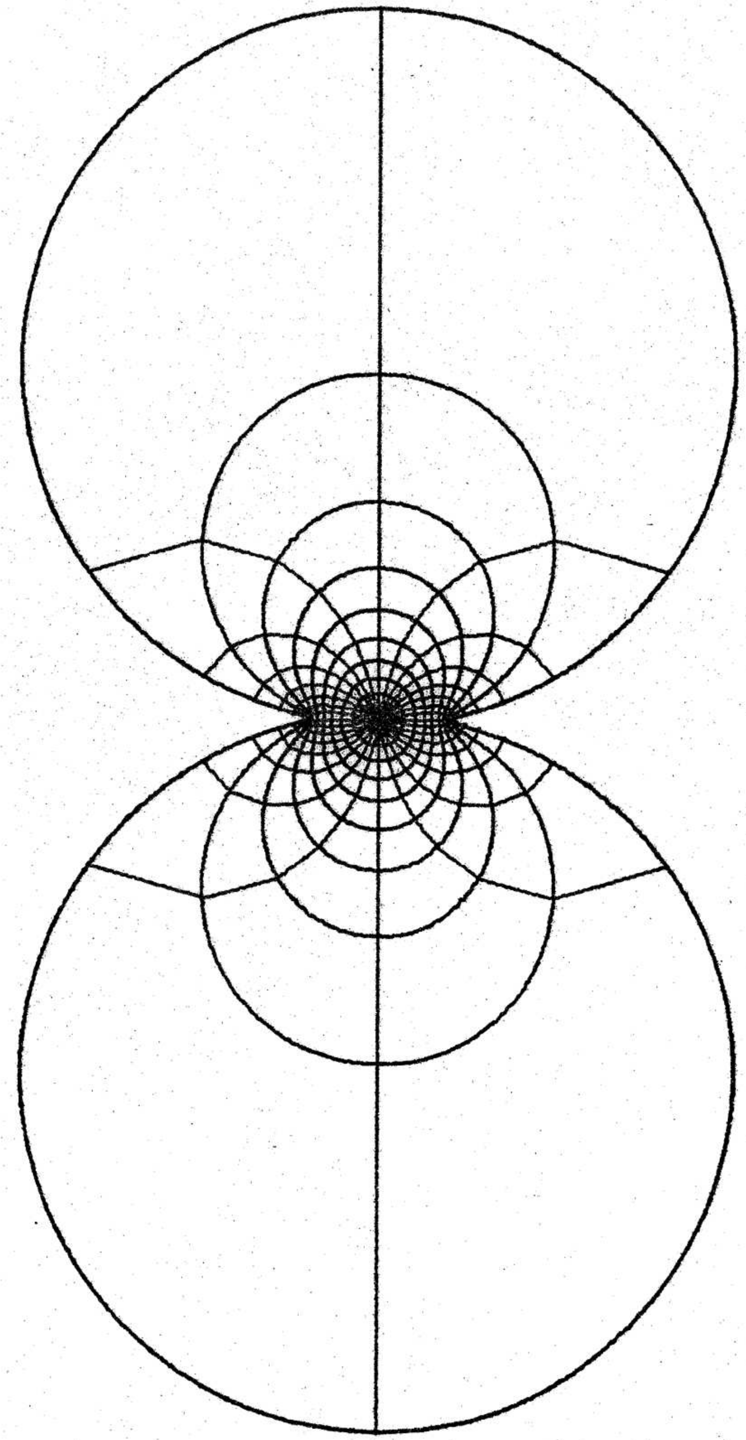
(drawing M. Gutknecht 18.12.1975)

Sketch of Proof: (Thesis §21; Th. Abelscher Funktionen 1857)

1. Place $\log(z - z_0)$;
2. correct Re $\log|z - z_0| - u(x, y)$,
u harmonic, boundary val. $\to 0$



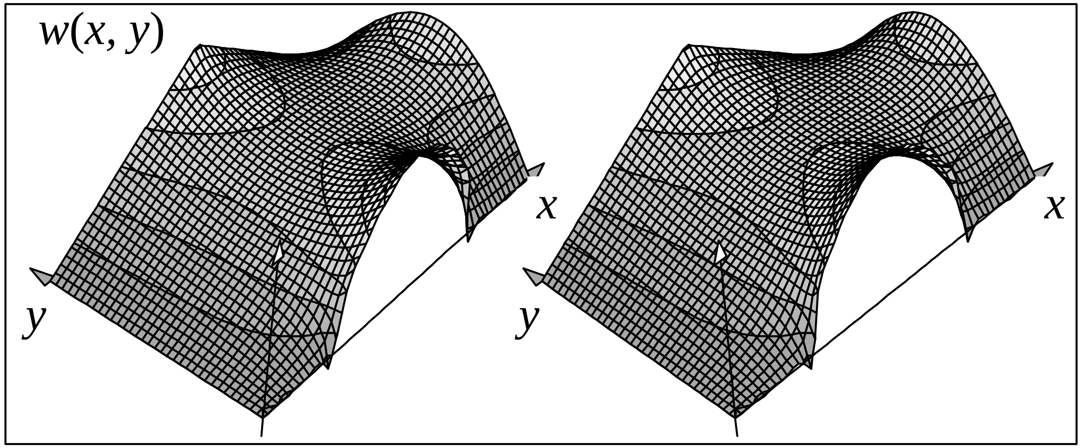
$\xleftarrow[f]{}$



3. complete $-iv(x, y)$ to holom. fcn.
4. $f = \exp\left(\log(z - z_0) - u - iv\right)$.

Riemann's Challenge:

Find harmonic functions $\Delta u = 0$ on any domain Ω with prescribed boundary conditions $u = F$ for $(x, y) \in \partial\Omega$?



easy for rectangle and circle (Fourier, Poisson), but difficult for arbitrary domains...

Riemann's Audacious "Proof":

> "Hierzu kann in vielen Fällen . . . ein Princip dienen, welches Dirichlet zur Lösung dieser Aufgabe für eine der Laplace'schen Differentialgleichung genügende Function . . . in seinen Vorlesungen . . . seit einer Reihe von Jahren zu geben pflegt." (Riemann *Crelle J.* 1857, *Werke* p. 97)

Idea. For all functions defined on a given domain Ω with the prescribed boundary values, the integral

$$J(u) = \iint_\Omega \frac{1}{2}\left(u_x^2 + u_y^2\right)\, dx\, dy \quad \text{is always} > 0.$$

Choose among these functions the one for which this integral is minimal !

(see citation; from here originates the name "Dirichlet Principle" and "Dirichlet boundary conditions").

Euler's "World" Upside Down !!

Differential Equation $\Delta u = 0$

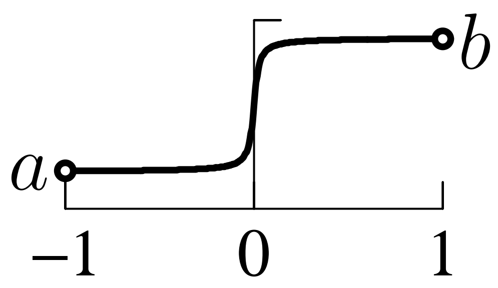
Dirichlet's Principle

Variational Problem $\int (u_x^2 + u_y^2)\,dx\,dy = \textbf{min}!$

Weierstrass' Critics: (1869, Werke 2, p. 49)

$$\int_{-1}^{1} (x \cdot y')^2 \, dx = \text{min!} \qquad y = \frac{a+b}{2} + \frac{b-a}{2} \, \frac{\arctan \frac{x}{\epsilon}}{\arctan \frac{1}{\epsilon}}$$

$$y(-1) = a, \;\; y(1) = b$$



"Die Dirichlet'sche Schlussweise führt also in dem betrachteten Falle offenbar zu einem falschen Resultat."

Riemann's Answer to Weierstrass:
"... meine Existenztheoreme sind trotzdem richtig".

(see F. Klein, *Entw. Math. 19. Jahrh.*, p. 264).

and Helmholtz: "Für uns Physiker bleibt das Dirichletsche Prinzip ein Beweis".

... but not for most mathematicians ...

Existence Proof without Dirichlet Principle:

H.A. Schwarz (1870, Crelle 74, 1872)

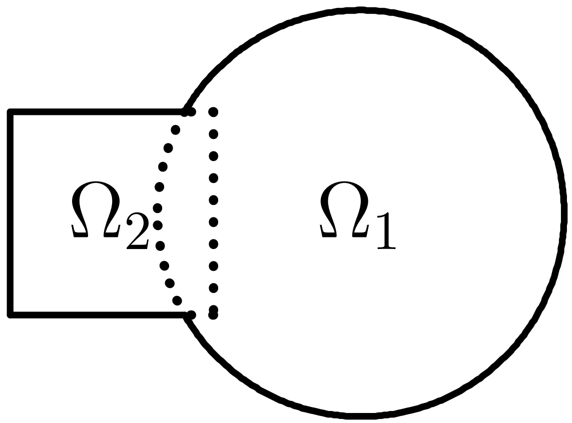
Solve alternatively in Ω_1 and Ω_2

new boundary values on dotted curves

$\Rightarrow$ converges

add third, fourth domain etc.

"Alternating method"



Rehabilitation of Dirichlet Principle:

C. Arzelà ("primo tentativo" 1897);

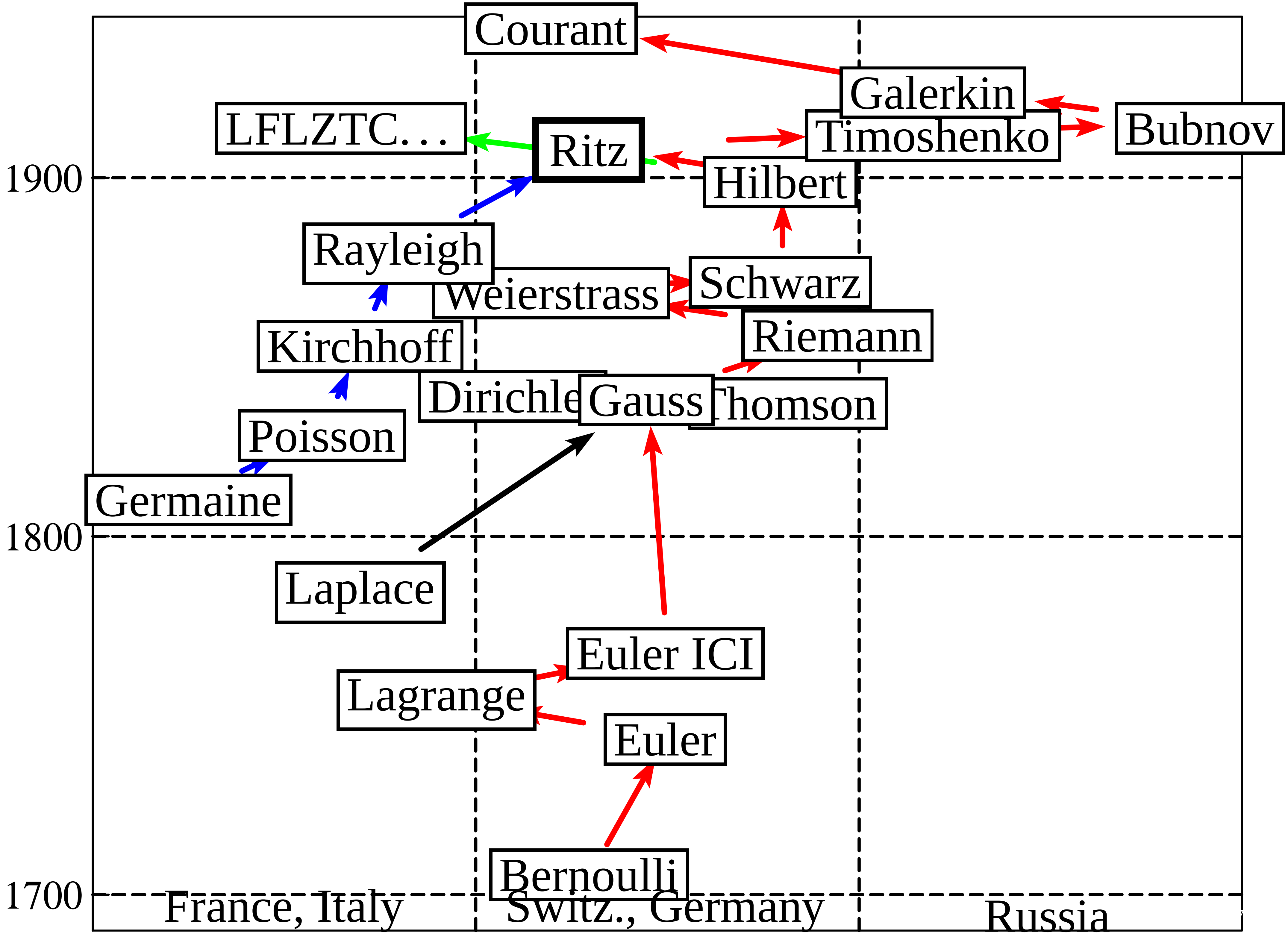
D. Hilbert ("trionfare" 1901, Annalen 1904, Crelle J. 1905):

"... eine besondere Kraftleistung beweisender Mathematik, ..."
(F. Klein, *Entw. Math. 19. Jahrh.*, p. 266).

Hilbert: "Das Dirichletsche Prinzip verdankte seinen Ruhm der anziehenden Einfachheit seiner mathematischen Grundidee, dem unleugbaren Reichtum der möglichen Anwendungen ... und der ihm innewohnenden Überzeugungskraft."

"Mittlerweile war das verachtete und scheintote *Dirichlet*sche Prinzip durch Hilbert wieder zum Leben erweckt worden;..."
(Hurwitz-Courant *Funktionentheorie*, Springer Grundlehren 3, p. 392)

$\Rightarrow$ Levi, Fubini, Lebesgue, Zaremba, Tonelli, Courant . . .



Courant
Galerkin
Bubnov
LFLZTC...
Ritz
Timoshenko
1900
Hilbert
Rayleigh
Schwarz
Weierstrass
Riemann
Kirchhoff
Dirichlet
Gauss
Thomson
Poisson
Germaine
1800
Laplace
Euler ICI
Lagrange
Euler
Bernoulli
1700
France, Italy
Switz., Germany
Russia

Another Problem: The Elastic Plate.

(S. Germaine 1811/13/15, corr. J.L. Lagrange, S.D. Poisson 1829, Kirchhoff (Crelle J. 40, 1850, p. 51-88))

Problem from the Academy of Paris for 1907:

Preisaufgaben der Académie des Sciences de Paris aus der angewandten Mathematik und Physik.

Für 1907. [1]

Prix Vaillant (4000 fr.): Perfectionner en un point important le problème d'Analyse relatif à l'équilibre des plaques élastiques encastrées, c'est-à-dire le problème de l'intégration de l'équation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f(x, y)$$

avec les conditions que la fonction u et sa dérivée suivant la normale au contour de la plaque soient nulles. Examiner plus spécialement le cas d'un contour rectangulaire. — Les Mémoires devront être envoyés au Secrétariat avant le 1er janvier 1907.

Ritz, had worked with many of such problems in his thesis trying to explain the Balmer series in spectroscopy (1902).

Ritz's method ...

Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik.

Von Herrn *Walter Ritz* in Göttingen.

... transforms the problem into a variational problem (“wie man ohne weiteres einsieht”) ...

$$\varDelta\varDelta w \equiv \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = f(x,y),$$

$$\Downarrow$$

$$J = \iint_R \left[\frac{1}{2}(\varDelta w)^2 - f(x,y)\,w\right] dS,$$

“Es ist ... J die potentielle Energie (... der *Kirchhoff* sche Ausdruck...)”

Ritz's method ...

... and approaches the solution globally by a linear combination of well chosen basis functions $\psi_1, \psi_2, \psi_3, \ldots$

$$J = \iint_R \left[\frac{1}{2}(\Delta w)^2 - f(x, y)\, w\right] dS,$$

$$\Downarrow$$

$$w_m = a_1 \psi_1 + a_2 \psi_2 + \cdots + a_m \psi_m$$

$$\Downarrow$$

$$J_m = \iint_R \left[\frac{1}{2}(\Delta w_m)^2 - f w_m\right] dS,$$

... transforming the problem into a finite dimensional minimization problem.

Explicitly:

$$J_m = \iint_R \left[\frac{1}{2} (\Delta w_m)^2 - f w_m \right] dS,$$

$$a_{ij} = \int_{-1}^{1} \int_{-1}^{1} \Delta\psi_i \cdot \Delta\psi_j \, dx \, dy \,, \quad b_i = \int_{-1}^{1} \int_{-1}^{1} \psi_i \cdot f \, dx \, dy \,.$$

$$J_m = \frac{1}{2} \sum_{i,j=1}^{m} a_{ij} \alpha_i \alpha_j - \sum_{i=1}^{m} b_i \alpha_i \quad \Rightarrow \text{min if} \quad \boxed{\sum_{j=1}^{m} a_{ij} \alpha_j = b_i}$$

First choice of basis functions (R = square $-1, \ldots 1$):

$$\psi_1(x,y) = (1-x^2)^2(1-y^2)^2$$
$$\psi_2(x,y) = (1-x^2)^2(1-y^2)^2(x^2+y^2)$$
$$\psi_3(x,y) = (1-x^2)^2(1-y^2)^2(x^4+y^4)$$
$$\psi_4(x,y) = (1-x^2)^2(1-y^2)^2x^2y^2$$
$$\psi_5(x,y) = (1-x^2)^2(1-y^2)^2(x^6+y^6)$$
$$\psi_6(x,y) = (1-x^2)^2(1-y^2)^2(x^4y^2+x^2y^4)\ldots$$

... leading to the system

$$\begin{array}{rrrrrrcr}
53.4988 & 9.7271 & 2.1339 & 0.5404 & 0.6374 & 0.3048 & = & 1.1378 \\
9.7271 & 21.9482 & 10.9187 & 1.7459 & 5.9936 & 1.2597 & = & 0.3251 \\
2.1339 & 10.9187 & 9.6331 & 0.9095 & 7.1004 & 1.0177 & = & 0.1084 \\
0.5404 & 1.7459 & 0.9095 & 0.4724 & 0.5147 & 0.3955 & = & 0.0232 \\
0.6374 & 5.9936 & 7.1004 & 0.5147 & 6.3226 & 0.7312 & = & 0.0493 \\
0.3048 & 1.2597 & 1.0177 & 0.3955 & 0.7312 & 0.4425 & = & 0.0155
\end{array}$$

$$
\begin{array}{rrrrrrcr}
\color{red}{53.4988} & 9.7271 & 2.1339 & 0.5404 & 0.6374 & 0.3048 & = & \color{red}{1.1378} \\
9.7271 & 21.9482 & 10.9187 & 1.7459 & 5.9936 & 1.2597 & = & 0.3251 \\
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0.3048 & 1.2597 & 1.0177 & 0.3955 & 0.7312 & 0.4425 & = & 0.0155
\end{array}
$$

“Cramer’s Rule” (Maclaurin 1748, Cramer 1750)

Then ſhall $z = \dfrac{aep - ahn + dhm - dbp + gbn - gem}{aek - ahf + dhc - dbk + gbf - gec}$.

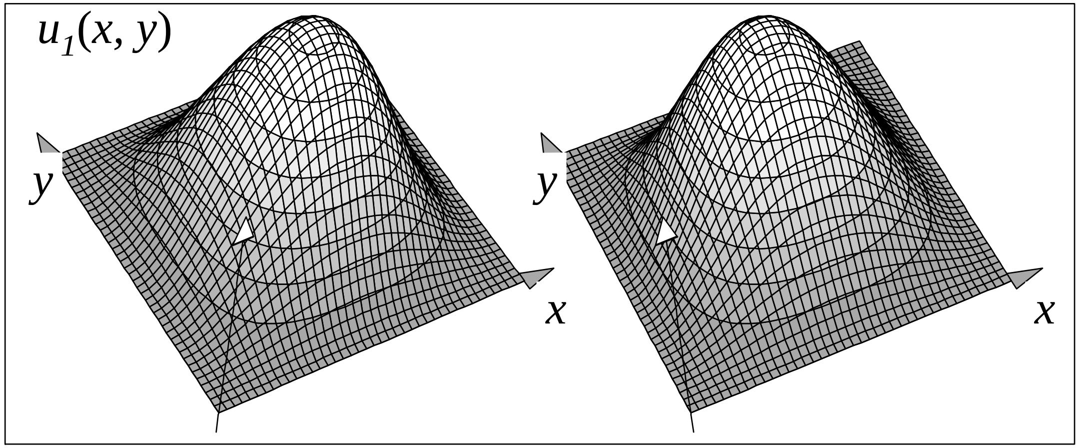
$$z = \frac{A^1Y^2X^3 - A^1Y^3X^2 - A^2Y^1X^3 + A^2Y^3X^1 + A^3Y^1X^2 - A^3Y^2X^1}{Z^1Y^2X^3 - Z^1Y^3X^2 - Z^2Y^1X^3 + Z^2Y^3X^1 + Z^3Y^1X^2 - Z^3Y^2X^1}$$

would require the computation of 5040 such terms.

However, because system is diagonally dominant, already

$$53.4988\,\alpha_1 = 1.1378 \qquad \Rightarrow \qquad \alpha_1 = \frac{1.1378}{53.4988} = 0.02127$$

gives an acceptable solution:



$m = 1$, errmax$= 0.04$

For higher precision Ritz computes iteratively

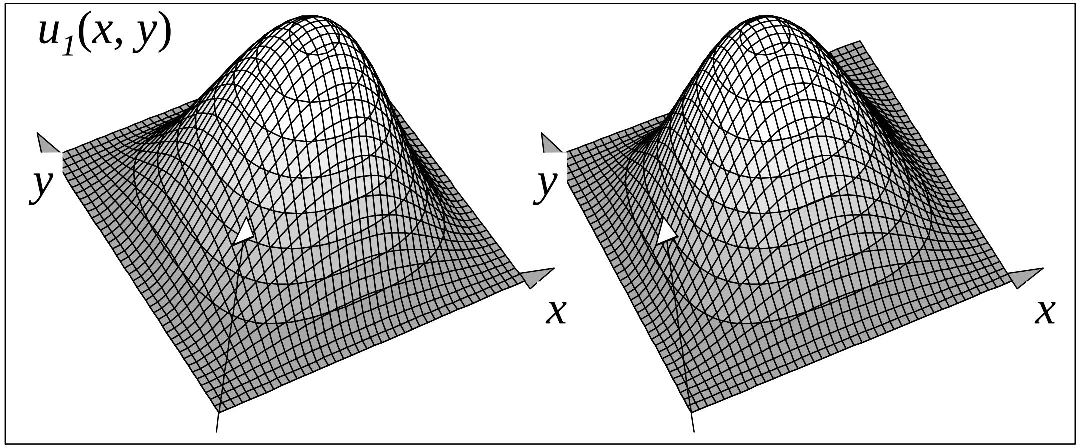
53.4988						=	1.1378
9.7271	21.9482					=	0.3251
2.1339	10.9187	9.6331				=	0.1084
0.5404	1.7459	0.9095	0.4724			=	0.0232
0.6374	5.9936	7.1004	0.5147	6.3226		=	0.0493
0.3048	1.2597	1.0177	0.3955	0.7312	0.4425	=	0.0155

and repeat ...

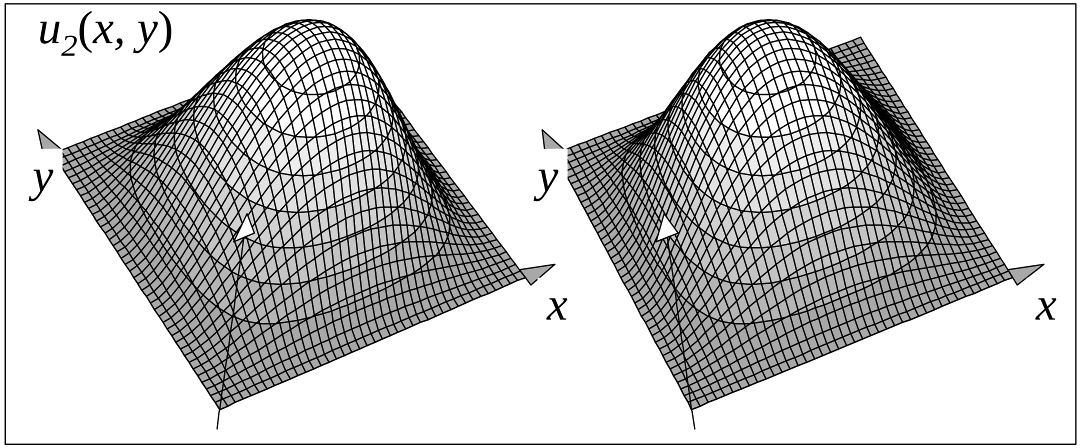
53.4988	9.7271	2.1339	0.5404	0.6374	0.3048	=	1.1378
9.7271	21.9482	10.9187	1.7459	5.9936	1.2597	=	0.3251
2.1339	10.9187	9.6331	0.9095	7.1004	1.0177	=	0.1084
0.5404	1.7459	0.9095	0.4724	0.5147	0.3955	=	0.0232
0.6374	5.9936	7.1004	0.5147	6.3226	0.7312	=	0.0493
0.3048	1.2597	1.0177	0.3955	0.7312	0.4425	=	0.0155

“wozu der Rechenschieber angewandt werden kann...
Eine direkte Lösung durch Determinanten würde 5stellige Logarithmentafeln erfordern”.

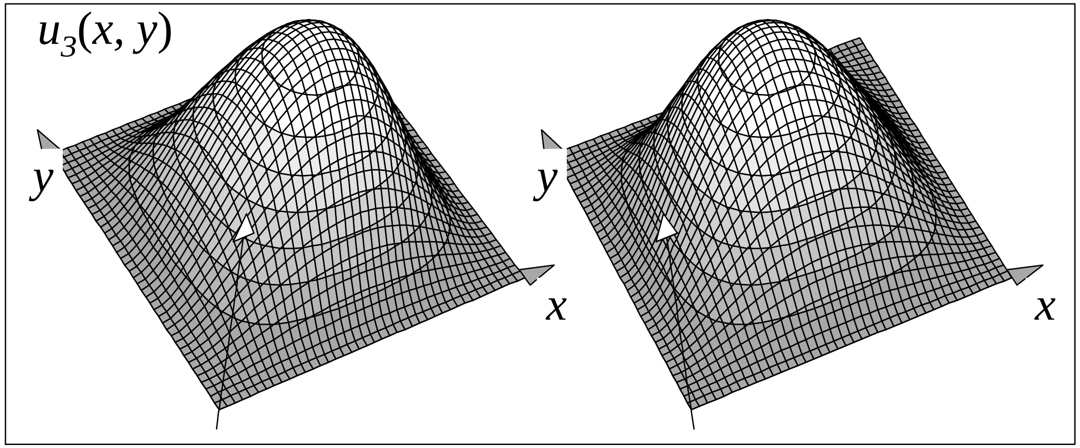
$$\alpha_1 = 0.020247 \quad \alpha_2 = 0.005209 \quad \alpha_3 = 0.000284$$
$$\alpha_4 = 0.006119 \quad \alpha_5 = -0.000019 \quad \alpha_6 = 0.000116$$



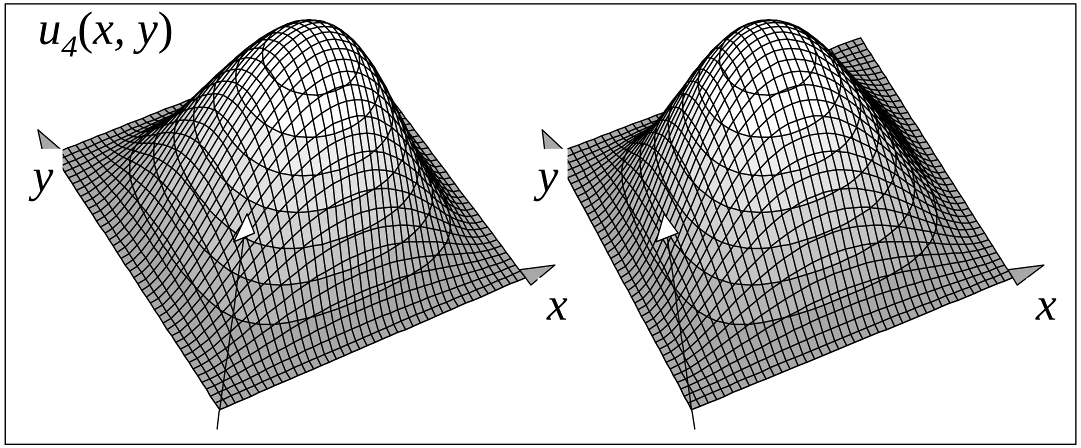
$m = 1$, errmax$= 0.04$



$m = 2$, errmax$= 0.003$



$m = 4$, errmax$= 0.000017$



$m = 6$

... beautiful in theory — but not in practice ...

$$
\begin{aligned}
\Delta\psi_6 = \; & 8\,x^2\left(1-y^2\right)^2\left(x^4y^2+x^2y^4\right) \\
& -8\,\left(1-x^2\right)\left(1-y^2\right)^2\left(4\,x^3y^2+2\,xy^4\right)x \\
& -4\,\left(1-x^2\right)\left(1-y^2\right)^2\left(x^4y^2+x^2y^4\right) \\
& +\left(1-x^2\right)^2\left(1-y^2\right)^2\left(12\,x^2y^2+2\,y^4\right) \\
& +8\,\left(1-x^2\right)^2y^2\left(x^4y^2+x^2y^4\right) \\
& -8\,\left(1-x^2\right)^2\left(1-y^2\right)\left(2\,x^4y+4\,x^2y^3\right)y \\
& -4\,\left(1-x^2\right)^2\left(1-y^2\right)\left(x^4y^2+x^2y^4\right) \\
& +\left(1-x^2\right)^2\left(1-y^2\right)^2\left(2\,x^4+12\,x^2y^2\right)
\end{aligned}
$$

must compute $\int_{-1}^{1}\int_{-1}^{1}\Delta\psi_6\cdot\Delta\psi_6\,dx\,dy=\frac{3052404736}{6898776885}$ from this ...
and 20 others ...

Dirichlet's Principle.

$$\Delta u = 0 \qquad u|_{\partial R} = F$$

or, after a subtraction,

$$-\Delta u = f \qquad u|_{\partial R} = 0\ .$$

We insert

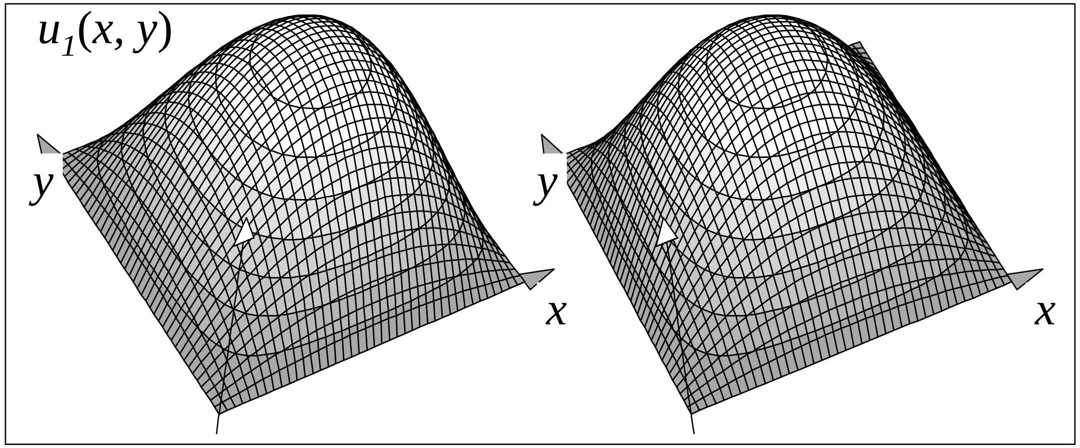
$$u_m = \alpha_1 \psi_1(x,y) + \ldots + \alpha_m \psi_m(x,y)$$

into

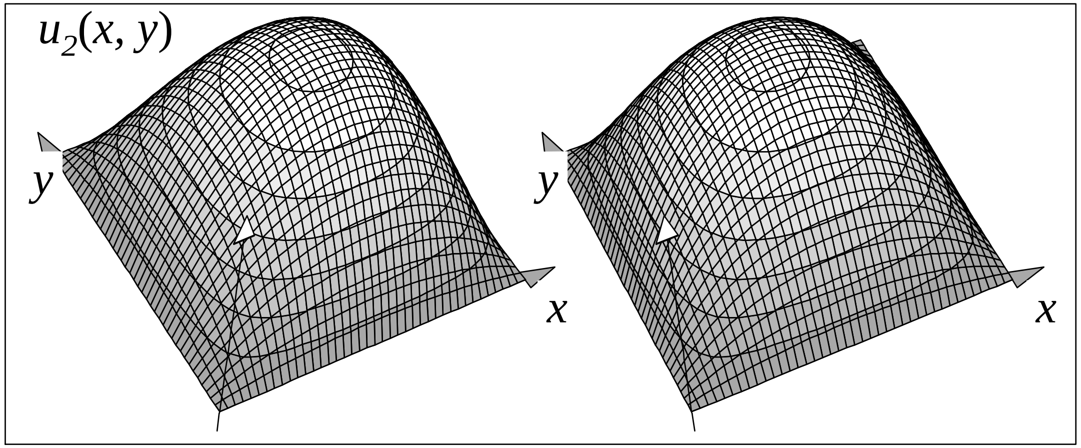
$$J = \iint_\Omega (\frac{1}{2}((\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2) - f \cdot u)\, dx\, dy = \text{min!}$$

modify the basis functions according to the new boundary condition ...

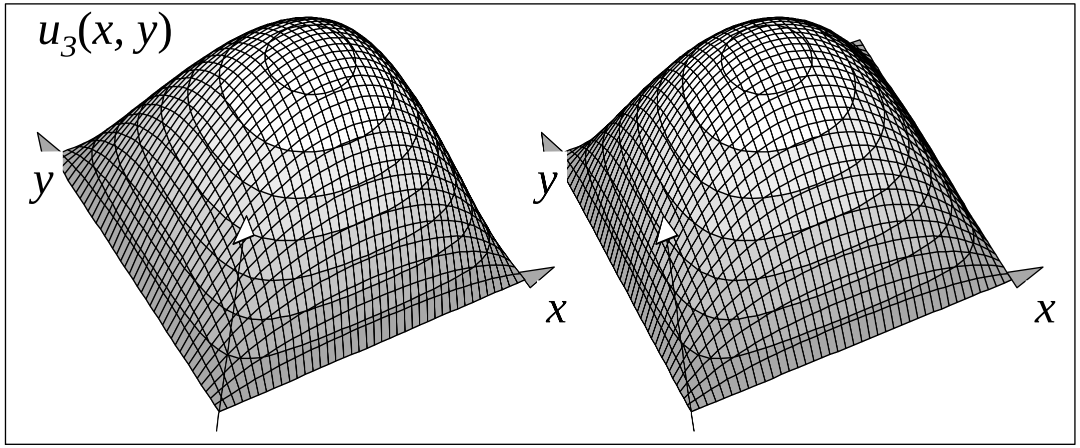
... with solution



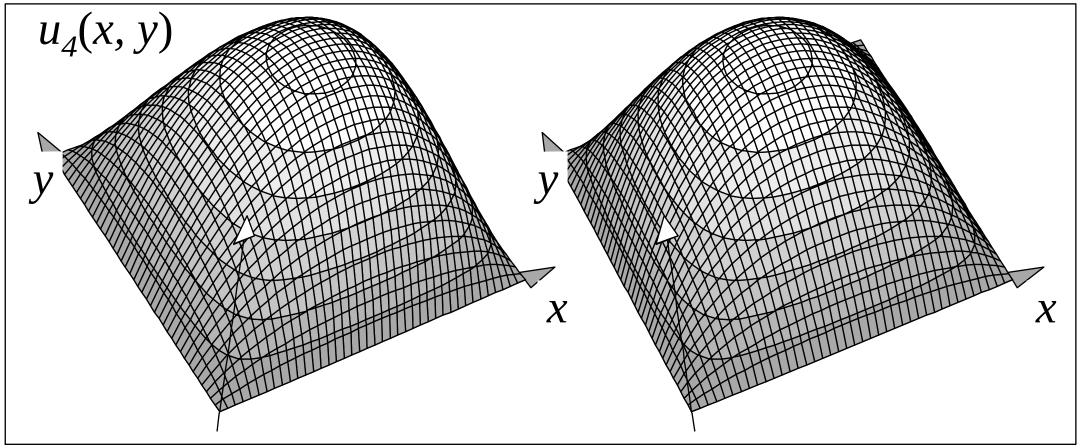
$m = 1$, errmax$= 0.05$



$m = 2$, errmax$= 0.01$

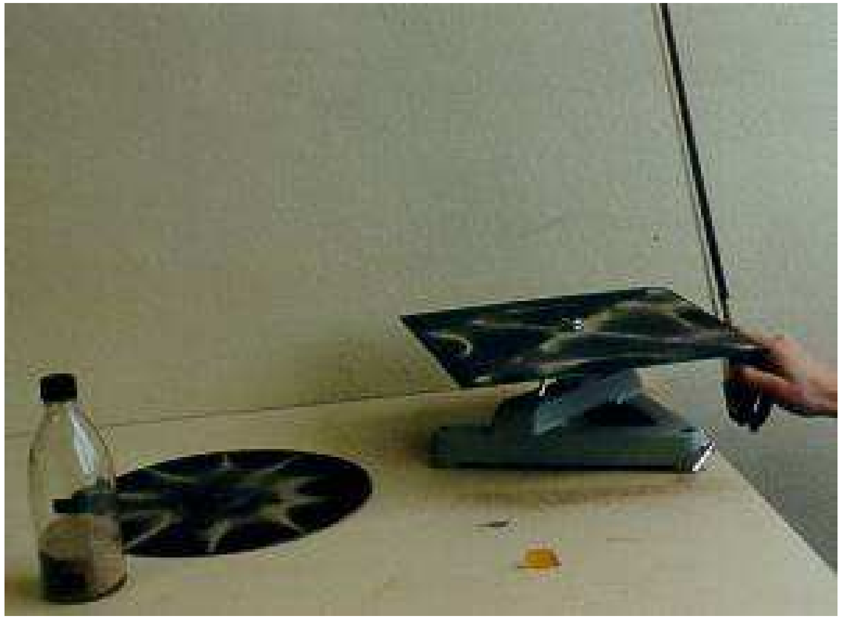


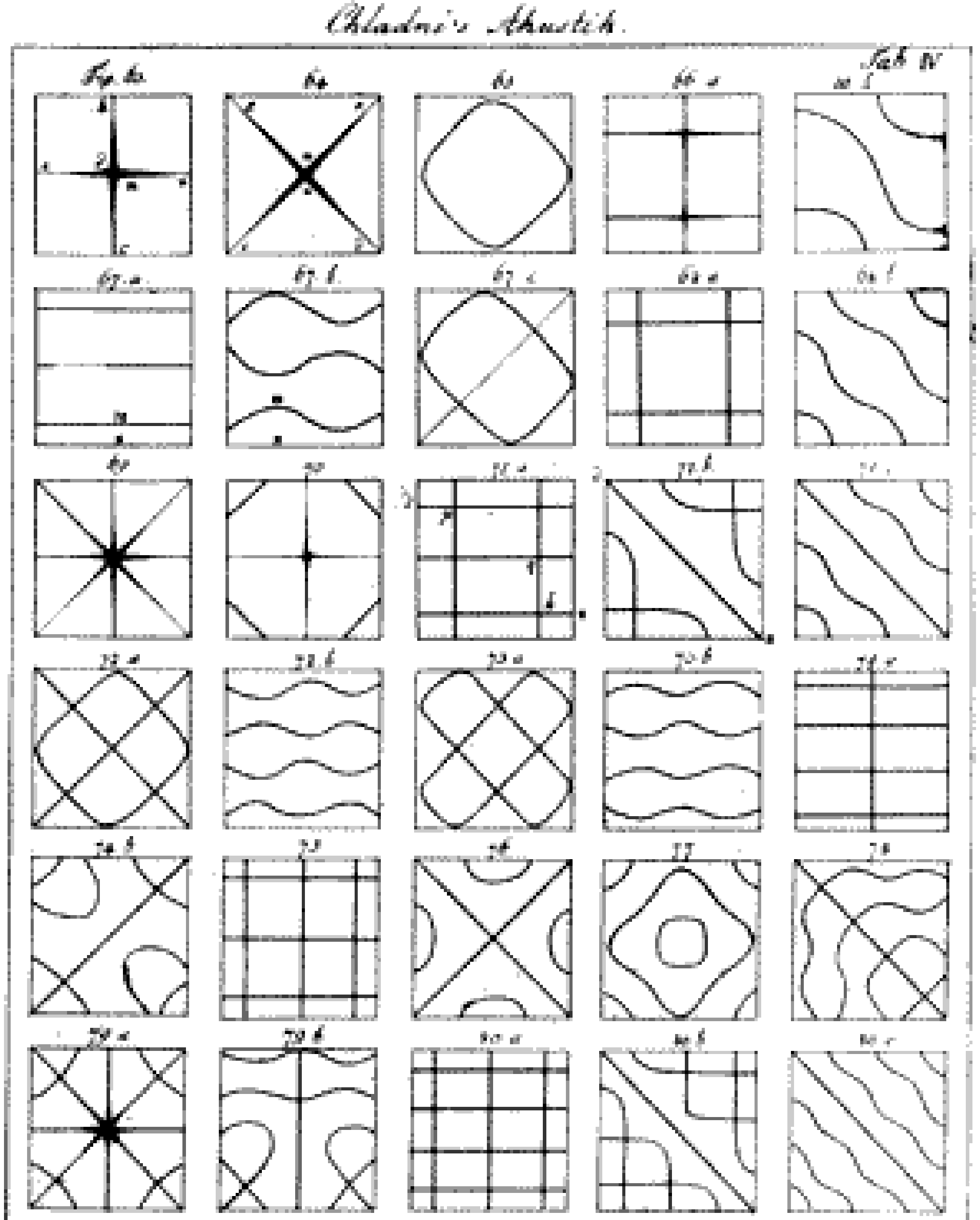
$m = 4$, errmax$= 0.0024$



$m = 6$

Ritz' Second Great Paper:
Ernst Florens Friedrich Chladni : Leipzig 1787.





Walther Ritz (1909): *Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern*

> "Die Differentialgleichungen und Randbedingungen für die transversalen Schwingungen ebener, elastischer Platten mit freien Rändern sind bekanntlich zuerst in teilweise unrichtiger Form von Sophie Germain und Poisson, in definitiver Gestalt aber von Kirchhoff im Jahre 1850 gegeben worden."

Chladni figures correspond to eigenpairs of the bi-harmonic operator

$$\Delta^2 w = \lambda w \qquad \text{in } \Omega := (-1,1)^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + (2-\mu)\frac{\partial^2 w}{\partial y^2}\right) = 0, \quad \frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2} = 0, \; x = \{-1,1\}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y^2} + (2-\mu)\frac{\partial^2 w}{\partial x^2}\right) = 0, \quad \frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2} = 0, \; y = \{-1,1\}$$

Here, μ is the elasticity constant.

Ritz' method now leads to an eigenvalue problem:

$u_m(y) = v_m$ gesetzt und

$$w = A_0 u_1 v_1 + A_1 (u_1 v_3 + v_1 u_3) + A_2 u_3 v_3 + A_3 (u_1 v_5 + u_5 v_1) + A_4 (u_3 v_5 + u_5 v_3) + A_5 u_5 v_5.$$

Das System (53) wird hier

$$(54) \quad \begin{cases} 0 = (13{,}95 - \lambda) A_0 - 32{,}08 A_1 + 18{,}60 A_2 + 32{,}08 A_3 - 37{,}20 A_4 + 18{,}60 A_5, \\ 0 = -16{,}04 A_0 + (411{,}8 - \lambda) A_1 - 120{,}0 A_2 - 133{,}6 A_3 + 166{,}8 A_4 + 140 A_5, \\ 0 = +18{,}60 A_0 - 240{,}0 A_1 + (1686 - \lambda) A_2 - 218{,}0 A_3 - 1134 A_4 + 330 A_5, \\ 0 = +16{,}04 A_0 - 133{,}6 A_1 + 109{,}0 A_2 + (2945 - \lambda) A_3 - 424 A_4 + 179 A_5, \\ 0 = -18{,}6 A_0 + 166{,}8 A_1 - 567 A_2 - 424 A_3 + (6303 - \lambda) A_4 - 1437 A_5, \\ 0 = +18{,}6 A_0 + 280 A_1 - 330 A_2 + 358 A_3 - 2874 A_4 + (13674 - \lambda) A_5. \end{cases}$$

2) How to solve the eigenvalue problem $K\vec{a} = \lambda\vec{a}$?

- setzen wir $A_0 = 1$, und in erster Annäherung $\lambda_0 = 13.95$. Dann ergeben die fünf letzten Gleichungen die übrigen A_i.
- Wir berechnen für die A_i eine erste Approximation, indem wir alle Glieder rechts vernachlässigen neben den Diagonalgliedern ...
- Ein oder zwei sukzessive Korrektionen genügen meist, um die vierte Stelle bis auf wenige Einheiten festzustellen.

$$
\begin{aligned}
0 &= (13{,}95 - \lambda) A_0 - 32{,}08 A_1 + 18{,}60 A_2 + 32{,}08 A_3 - 37{,}20 A_4 + 18{,}60 A_5, \\
0 &= -16{,}04 A_0 + (411{,}8 - \lambda) A_1 - 120{,}0 A_2 - 133{,}6 A_3 + 166{,}8 A_4 + 140 A_5, \\
0 &= +18{,}60 A_0 - 240{,}0 A_1 + (1686 - \lambda) A_2 - 218{,}0 A_3 - 1134 A_4 + 330 A_5, \\
0 &= +16{,}04 A_0 - 133{,}6 A_1 + 109{,}0 A_2 + (2945 - \lambda) A_3 - 424 A_4 + 179 A_5, \\
0 &= -18{,}6 A_0 + 166{,}8 A_1 - 567 A_2 - 424 A_3 + (6303 - \lambda) A_4 - 1437 A_5, \\
0 &= +18{,}6 A_0 + 280 A_1 - 330 A_2 + 358 A_3 - 2874 A_4 + (13674 - \lambda) A_5.
\end{aligned}
$$

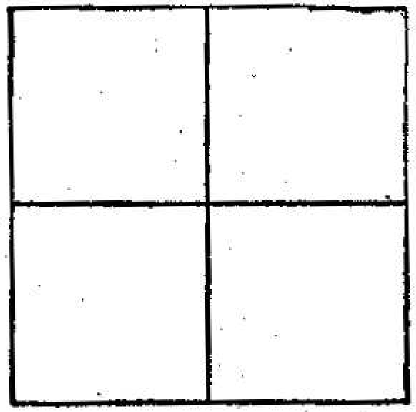
iterative linear algebra, iterative eigenvalue calculation ...

Ritz is very modern in 1909 !!

Chladni Figures Computed by Ritz

A. *Lösungen, die in x und y ungerade und symmetrisch sind.*

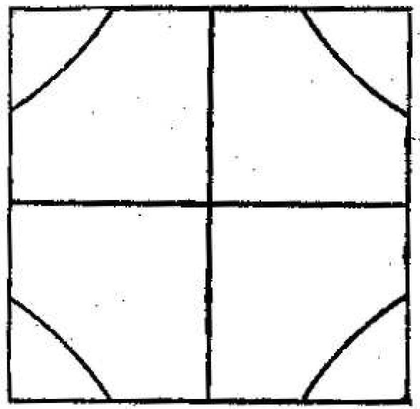
Fig. 1.



I. *Grundton.* $\lambda = 12,43 - 18,0\,\delta\mu.$

$$w = \mathbf{u}_1\mathbf{v}_1 + 0,0394(u_1 v_3 + v_1 u_3) - 0,0040\,u_3 v_3 - 0,0034(u_1 v_5 + u_5 v_1) + 0,0011(u_3 v_5 + u_5 v_3) - 0,0019\,u_5 v_5.$$

Fig. 2.



II. $\lambda = 378 - 57\,\delta\mu.$

$$w = -0,075\,u_1 v_1 + (\mathbf{u}_1\mathbf{v}_3 + \mathbf{u}_3\mathbf{v}_1) + 0,173\,u_3 v_3 + 0,045(u_1 v_5 + u_5 v_1) - 0,015(u_3 v_5 + u_5 v_3) - 0,029\,u_5 v_5.$$

Es ist

y beob. :	0,530	0,578	0,630	0,690	0,752	0,819	0,893

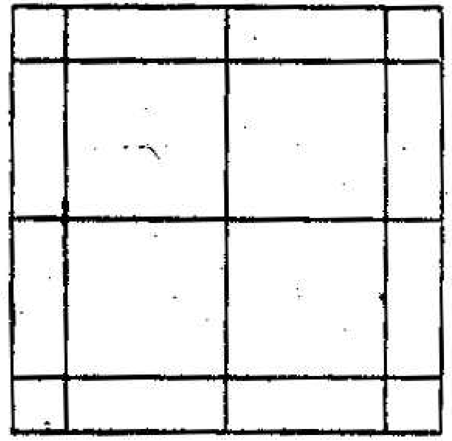
III. $\lambda = 1554.$

$$w = 0{,}009\, u_1 v_1 - 0{,}075 (u_1 v_3 + v_1 u_3) + \mathbf{u}_3 \mathbf{v}_3 - 0{,}057 (u_1 v_5 + u_5 v_1) + 0{,}121 (u_3 v_5 + u_5 v_3) - 0{,}007\, u_5 v_5.$$

Messungen fehlen.

R.

Fig. 3.



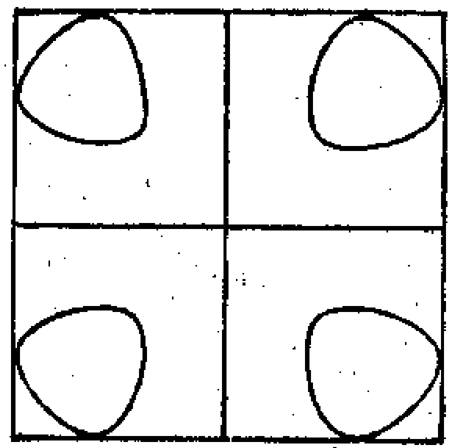
20

IV. $\lambda = 2945.$

$$w = u_1 v_5 + u_5 v_1.$$

Diese Figur fehlt bei Chladni.

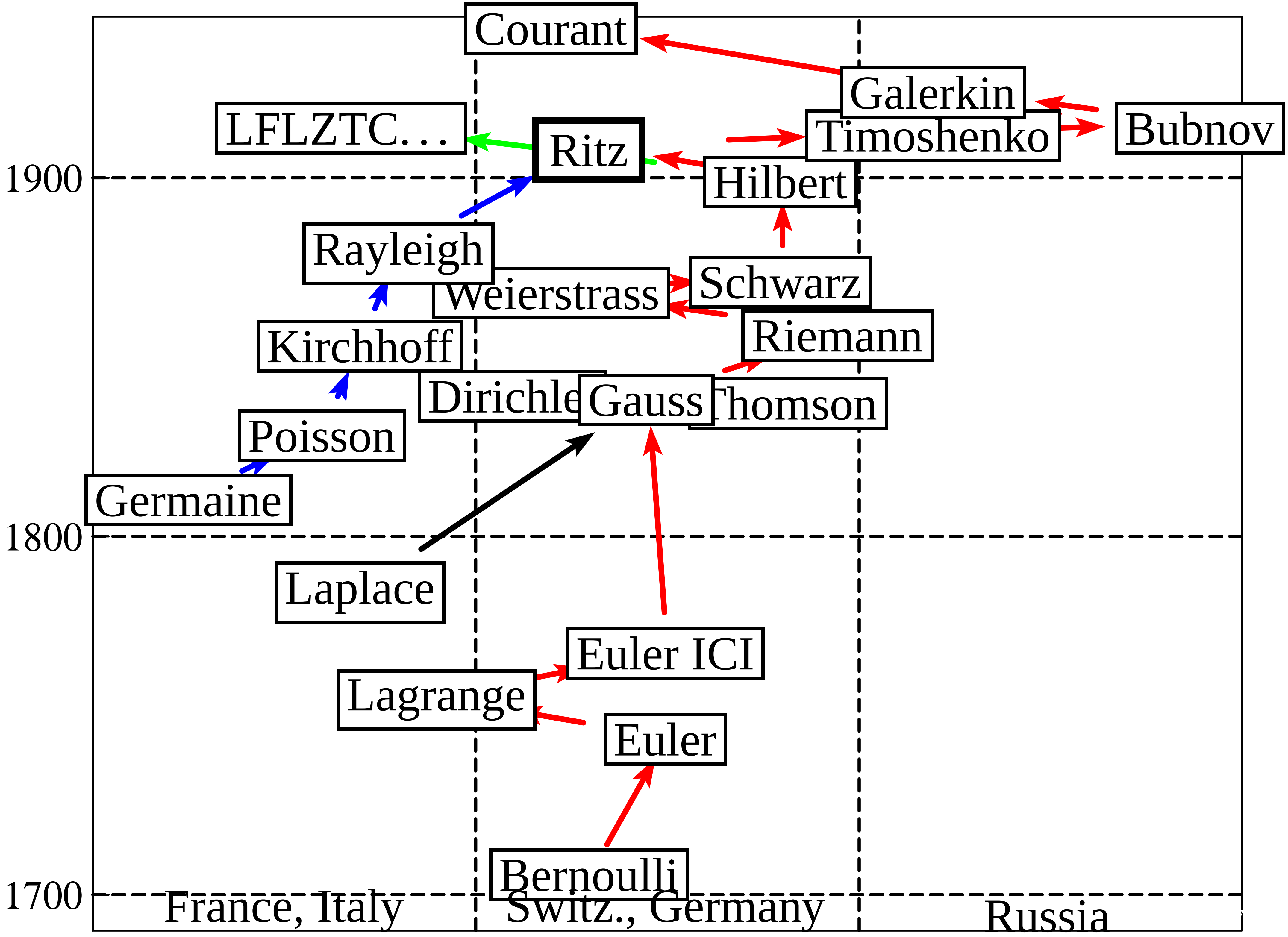
Fig. 4.



Frequency Table Computed by Ritz

Tabelle der Tonhöhen ($\mu = 0{,}225$).

HAUPTGLIEDER	λ	BER.	BEOB.	HAUPTGLIEDER	λ	BER.	BEOB.
$u_1 v_1$	12,43	G*	G	$u_3 v_4 \pm u_4 v_3$	3240	g_3+	fis_3
$u_0 v_2 - v_0 u_2$	26,40	d*	d	$u_5 v_2 \pm u_2 v_5$	3927	a_3+	gis_3+
$u_0 v_2 + v_0 u_2$	35,73	e*	e	$u_4 v_4$	5480	ais_3+	ais_3
$u_1 v_2 \pm u_2 v_1$	80,8	h*	h	$u_0 v_6 - u_6 v_0$	5500	c_4-	— [2]
$u_0 v_3 \pm u_3 v_0$	237,1	gis_1* +	gis_1+	$u_3 v_5 - u_5 v_3$	5570	c_4-	ais_3-
$u_1 v_1$	266,0	ais_1* —	ais_1* —	$u_0 v_6 + u_6 v_0$	5640	c_4-	— [2]
$u_1 v_3 - u_3 v_1$	316,1	h_1*	h_1	$u_1 v_6 \pm v_1 u_6$	6036	c_4+	c_4-
$u_1 v_3 + u_3 v_1$	378	cis_2*	cis_2	$u_3 v_5 + u_5 v_3$	6303	cis_4	c_4-
$u_2 v_3 \pm u_3 v_2$	746	fis_2* +	fis_2	$u_2 v_6 - u_6 v_2$	7310	d_4+	cis_4+
$u_0 v_4 - v_0 u_4$	886	gis_2	gis_2	$u_2 v_6 + u_6 v_2$	7840	dis_4-	d_4-
$u_0 v_4 + v_0 u_4$	941	gis_2+	gis_2+	$u_5 v_4 \pm u_4 v_5$	9030	e_4	dis_4
$u_1 v_4 \pm u_4 v_1$	1131	ais_2	ais_2-	$u_6 v_3 \pm u_3 v_6$	10380	f_4	e_4
$u_3 v_3$	1554	c_3+	c_3	$u_5 v_5$	13670	g_4+	fis_4+
$u_2 v_4 - u_4 v_2$	1702	d_3-	cis_3	$u_6 v_4 - u_4 v_6$	13840	g_4+	g_4+
$u_2 v_4 + u_4 v_2$	2020	dis_3	d_3	$u_6 v_4 + u_4 v_6$	15120	gis_4+	g_4+
$u_0 v_5 \pm v_0 u_5$	2500	f_3-	f_3-	$u_6 v_5 \pm u_5 v_6$	20400	h_4	ais_4-
$u_1 v_5 - v_1 u_5$	2713	fis_3	fis_3-	$u_6 v_6$	28740	d_5	— [2]


Courant
Galerkin
Bubnov
LFLZTC. . .
Ritz
Timoshenko
Hilbert
1900
Rayleigh
Weierstrass
Schwarz
Riemann
Kirchhoff
Dirichlet
Gauss
Thomson
Poisson
Germaine
1800
Laplace
Euler ICI
Lagrange
Euler
Bernoulli
1700
France, Italy
Switz., Germany
Russia

Ritz’ ideas ⇒ Russia: S.P. Timoshenko (Kiev 1910) :

ОБ УСТОЙЧИВОСТИ УПРУГИХ СИСТЕМ. ПРИМЕНЕНИЕ НОВОЙ МЕТОДЫ К ИССЛЕДОВАНИЮ УСТОЙЧИВОСТИ НЕКОТОРЫХ МОСТОВЫХ КОНСТРУКЦИЙ. [1])

Известия Киевского политехнического института, 1910, год 10, Отдел инженерной механики, книга 4, стр. 375—560. Отд. оттиск, Киев, тип. С. В. Кульженко, 1910, 188 стр. То же, 1911. Перевод на французский язык: Sur la stabilité des systèmes élastiques. Application d’une nouvelle méthode à la recherche de la stabilité de certaines parties constitutives des ponts. Annales des ponts et chauseés, 9th séries, 1913, tome 15, vol. 3, Mai — Juin, № 24, pp. 496—566; tome 16, vol. 4, Juillet — Août, № 39, pp. 73—132; tome 17, vol. 5, Septembre

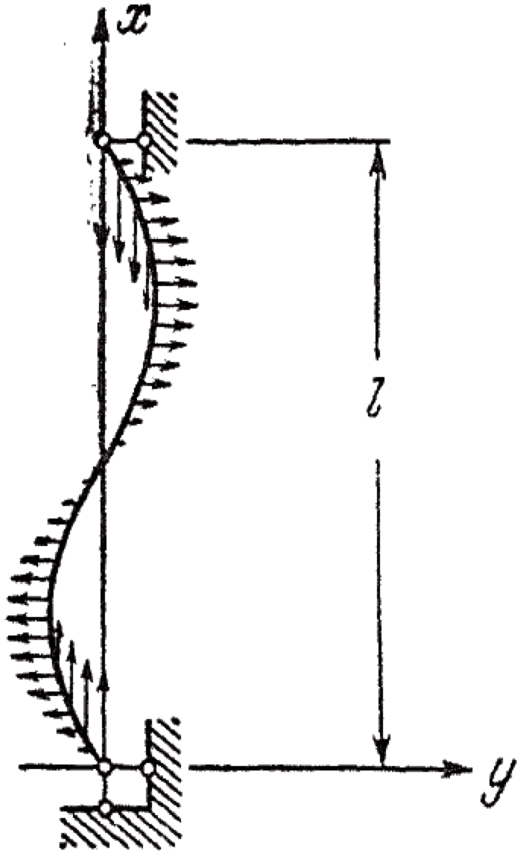
Nous ne nous arrêterons plus sur le côté mathématique de cette question: un ouvrage remarquable du savant suisse, M. Walter Ritz, . . .

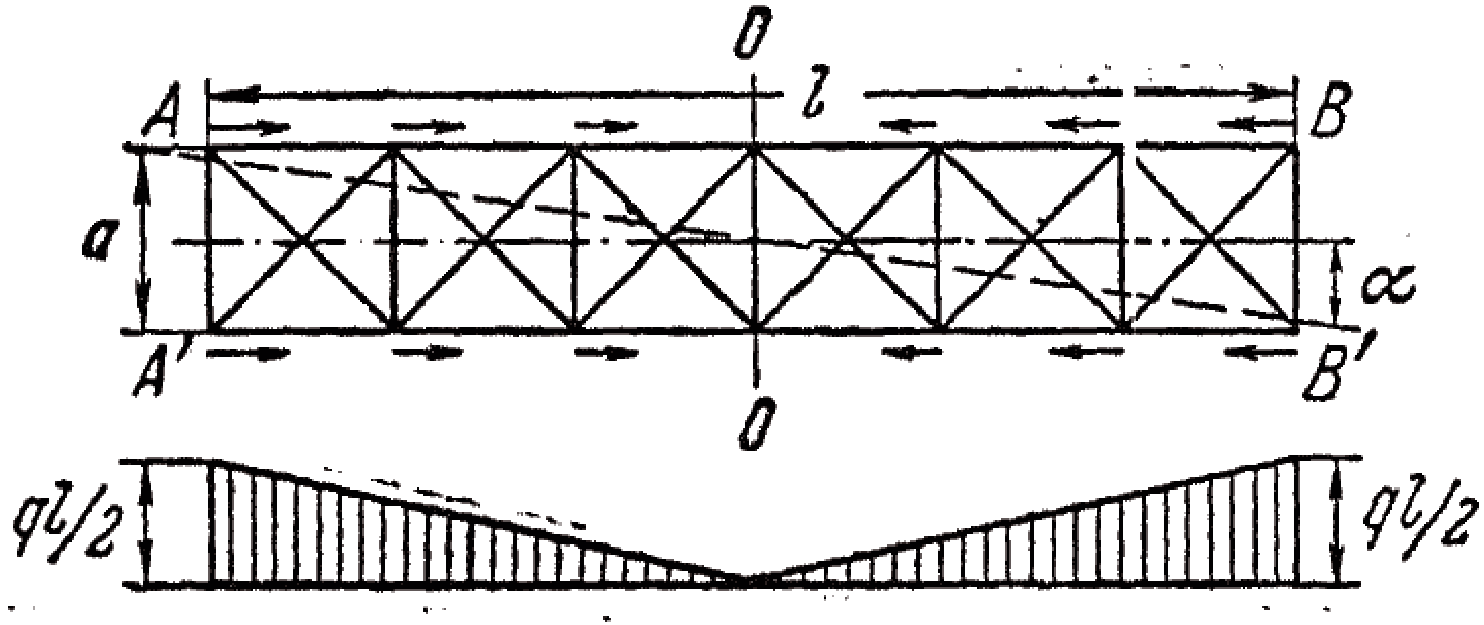
швейцарского ученого Вальтера Ритца

S.P. Timoshenko (1910) :

$$w = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \ldots$$

$$y = A\sin\frac{\pi x}{l} + A_1\sin\frac{2\pi x}{l} + A_2\sin\frac{3\pi x}{l} + \ldots$$





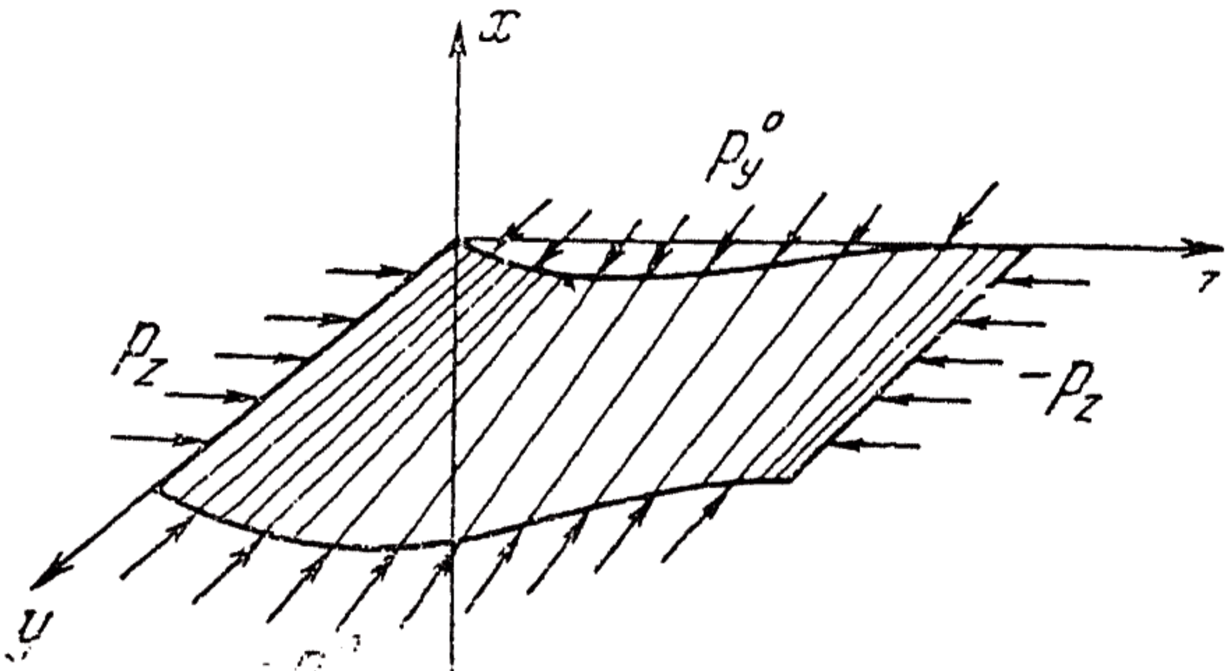
Ivan Bubnov (1872-1919)

СТРОИТЕЛЬНАЯ МЕХАНИКА КОРАБЛЯ[1]

[РАЗДЕЛЫ, ОТНОСЯЩИЕСЯ К ТЕОРИИ ПЛАСТИН]

Structural Mechanics of Shipbuilding

[Part concerning the theory of shells]



Boris Grigoryevich Galerkin (1871-1945)

СТЕРЖНИ И ПЛАСТИНКИ

РЯДЫ В НЕКОТОРЫХ ВОПРОСАХ УПРУГОГО РАВНОВЕСИЯ СТЕРЖНЕЙ И ПЛАСТИНОК*

(Петроград, 1915)

Beams and Plates

Series solution of some problems in elastic equilibrium of rods and plates

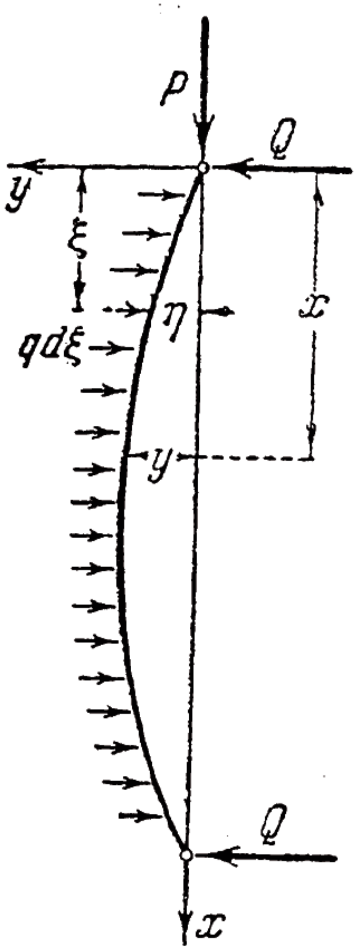
(Petrograd, 1915)

(Boris Grigoryevich Galerkin and the famous paper which is now quoted in the literature for the invention of the "Galerkin" method)

Из приближенных методов решений широкое применение получил в последнее время метод Ритца [6, 7]. Этот метод сводится вкратце к следующему.

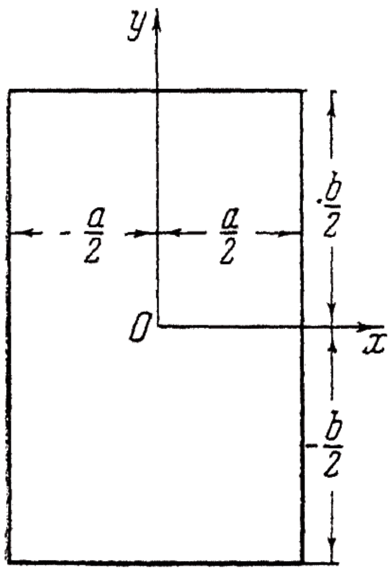
B.G. Galerkin (1915) :

$$w = \sum_{k=1}^{k=\infty} \sum_{n=1}^{n=\infty} A_{kn} \sin \frac{k\pi x}{a} \sin \frac{n\pi y}{b}$$



B. Изгиб пластинки

14. *Пластинка с закрепленными краями, опертая по наружному обводу.*



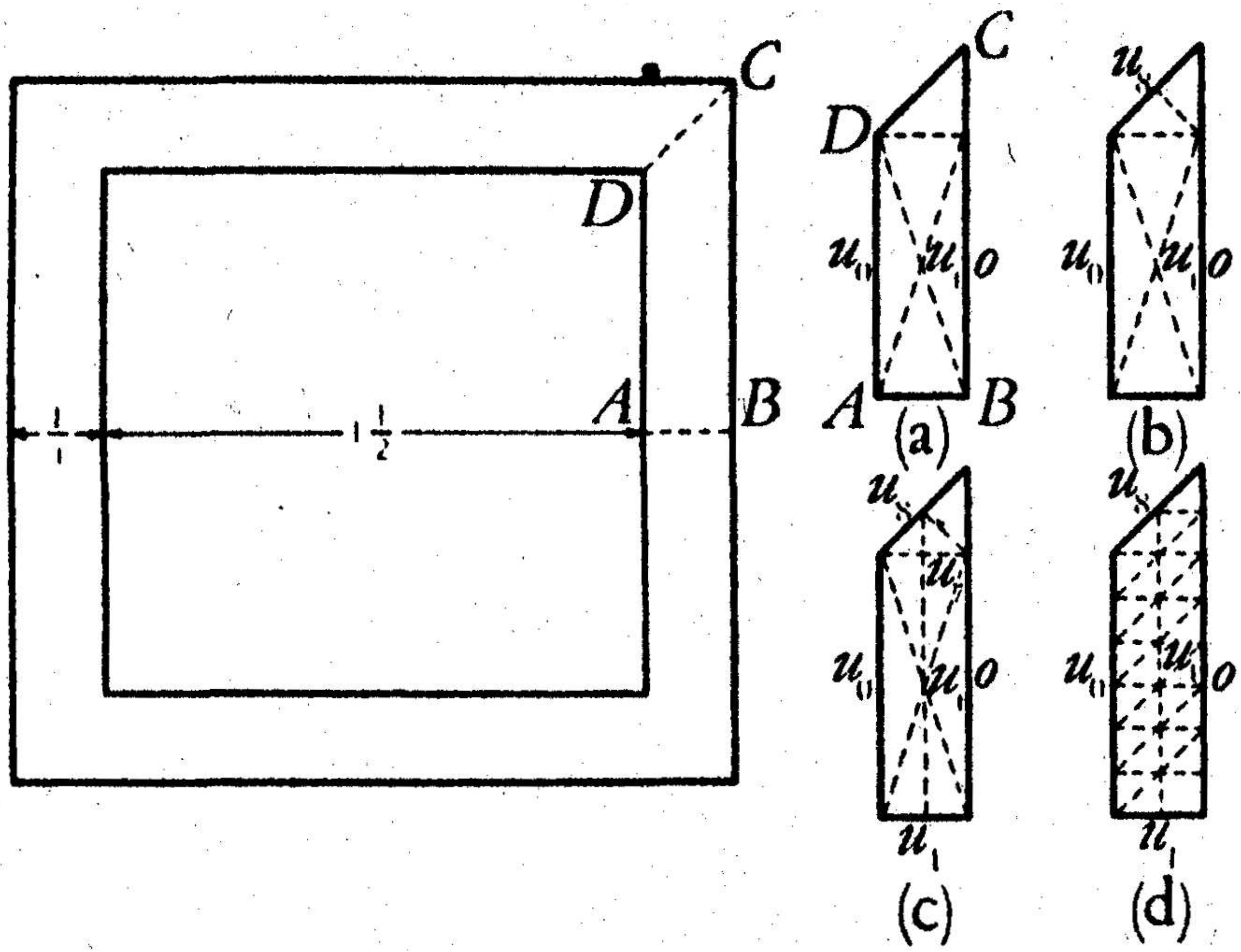
Мы рассматриваем здесь пластинку, опертую краями на неупругие опоры и нагруженную равномерно распределенной по поверхности нагрузкой p.

Начало координат взято в центре не изогнутой пластинки; оси координат направлены параллельно сторонам прямоугольника (фиг. 8).

За уравнение упругой поверхности берем

$$w = \sum_{k=2}^{k=\infty} \sum_{n=2}^{n=\infty} A_{kn} (a^2 - 4x^2)^k (b^2 - 4y^2)^n \qquad (17)$$

Richard Courant (address to the AMS, on May 3rd, 1941)

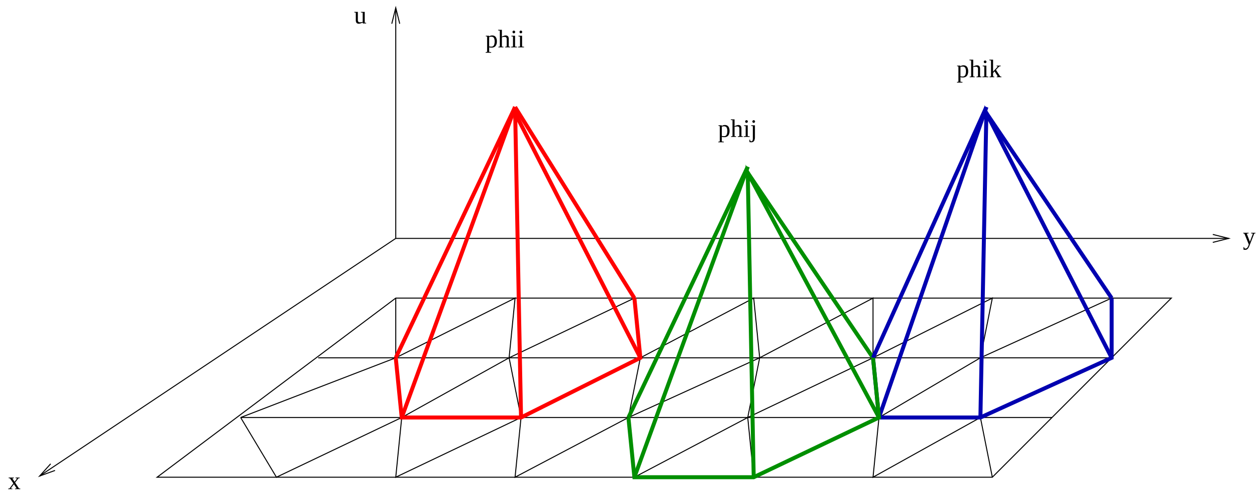


“At first, the theoretical interest in existence proofs dominated, and only much later were practical applications envisaged...

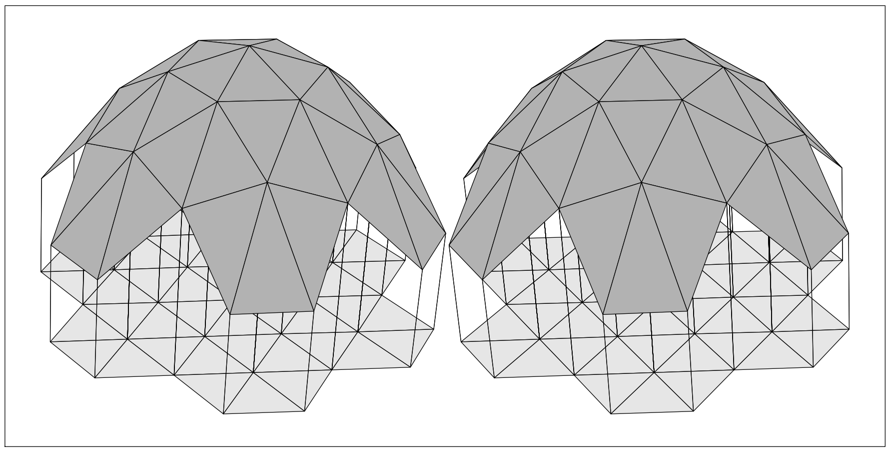
First mention of Finite Element Method:

[1]) Die wirkliche, für den bloßen Existenzbeweis unerhebliche Konstruktion solcher Minimalfolgen macht keinerlei prinzipielle Schwierigkeiten. Ist z. B. G ein ganz im Endlichen gelegener Bereich, begrenzt von Kurven C ohne mehrfache Punkte, so denken wir uns denselben mit einem noch von dem Index j abhängigen Dreiecksnetz T_j überdeckt, dessen Maschen mit wachsendem j immer enger werden. Wir betrachten nun nur solche Funktionen φ bzw. $\Phi = \varphi - S$, wo die Differenz $\varphi - \frac{x}{x^2+y^2}$ in jedem Dreieck von T_j eine lineare Funktion wird. Unter Φ_j verstehen wir diejenige unter den so entstehenden zu T_j konstruierten Funktionen, für welche $D[\Phi]$ einen möglichst kleinen Wert erhält. Diese Forderung $D[\Phi] = \text{Min.}$ ist jetzt ein Problem eines Minimums einer Funktion von einer endlichen Anzahl von Variablen, nämlich des Integrals, aufgefaßt in seiner Abhängigkeit von den Werten von φ in den Eckpunkten der Dreieckseinteilung; dieses Problem ist gewiß lösbar, und zwar, wie leicht ersichtlich, mittels linearer Gleichungen. Daß die so entstehenden Funktionen Φ_j wirklich eine Minimalfolge bilden, folgt sofort aus der unschwer beweisbaren Tatsache, daß man jede zulässige Funktion Φ und deren Dirichletsches Integral mit Hilfe unserer Konstruktion bei hinreichend großem j beliebig genau approximieren kann.

(Footnote in the first edition of the book by Hurwitz and Courant (1922) with first sketch of triangular FE method for proof of Riemann Mapping Theorem; removed in second edition) the base functions $\psi_i(x, y)$ (compact support):



and the linear combinations $\alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3 + \ldots$

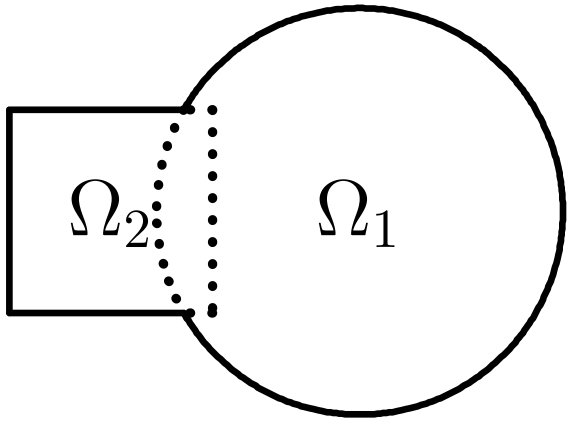


$\Rightarrow$ matrix A has many zeros and is easy to calculate.

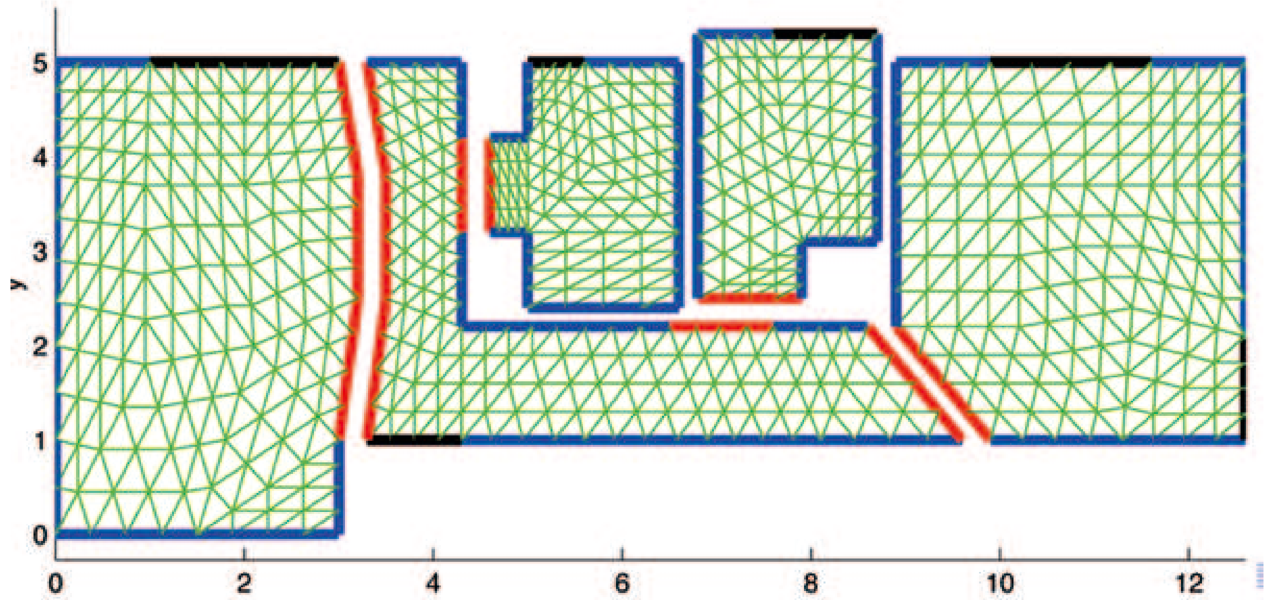
Difficulty: For large problems convergence of iterations for liner system **extremely slow.**

Remedy: inspired by

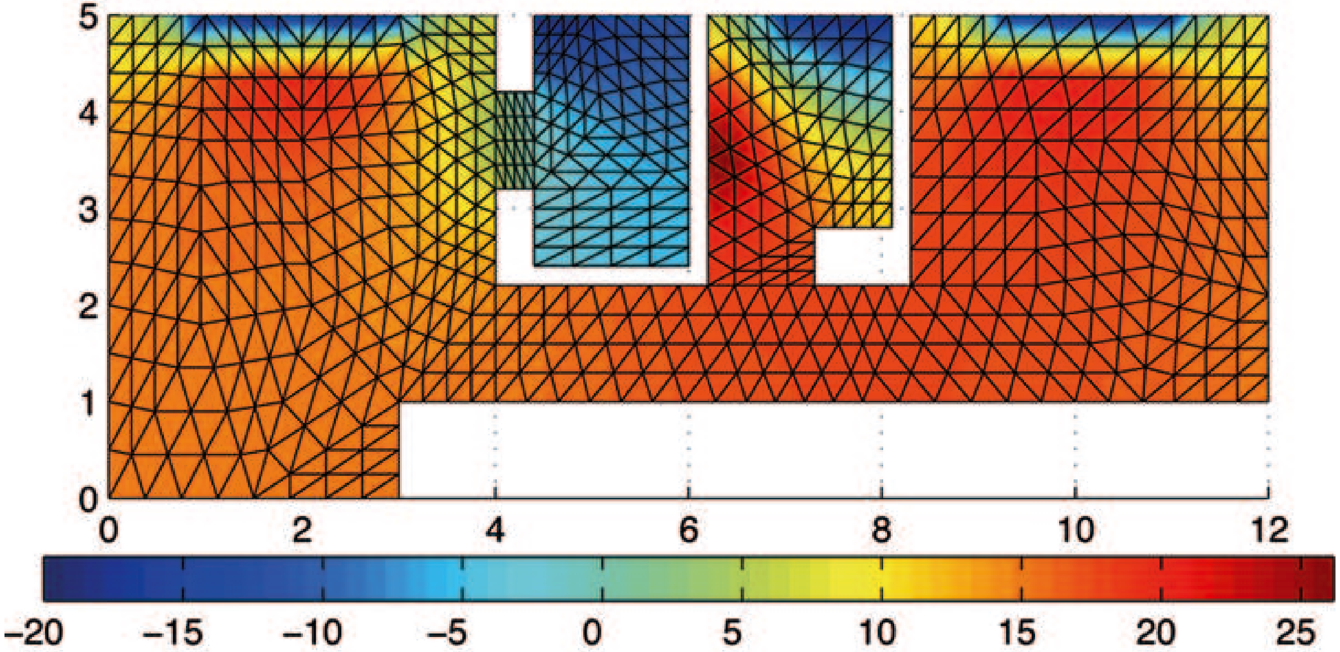
H.A. Schwarz (1870, Crelle 74, 1872)
Solve alternatively in Ω_1 and Ω_2
"Alternating method"

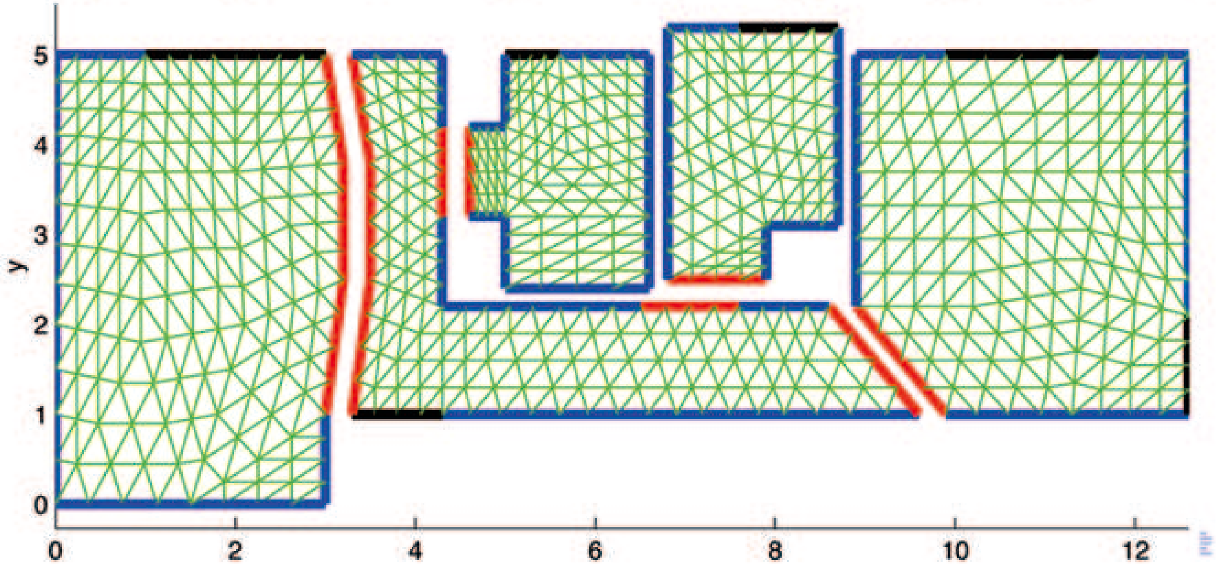


$\Rightarrow$ **Domain Decomposition Method**



... voilà ...





$\Delta u = f$ Ritz-Galerkin FE method Domain decomp.

Grazie !