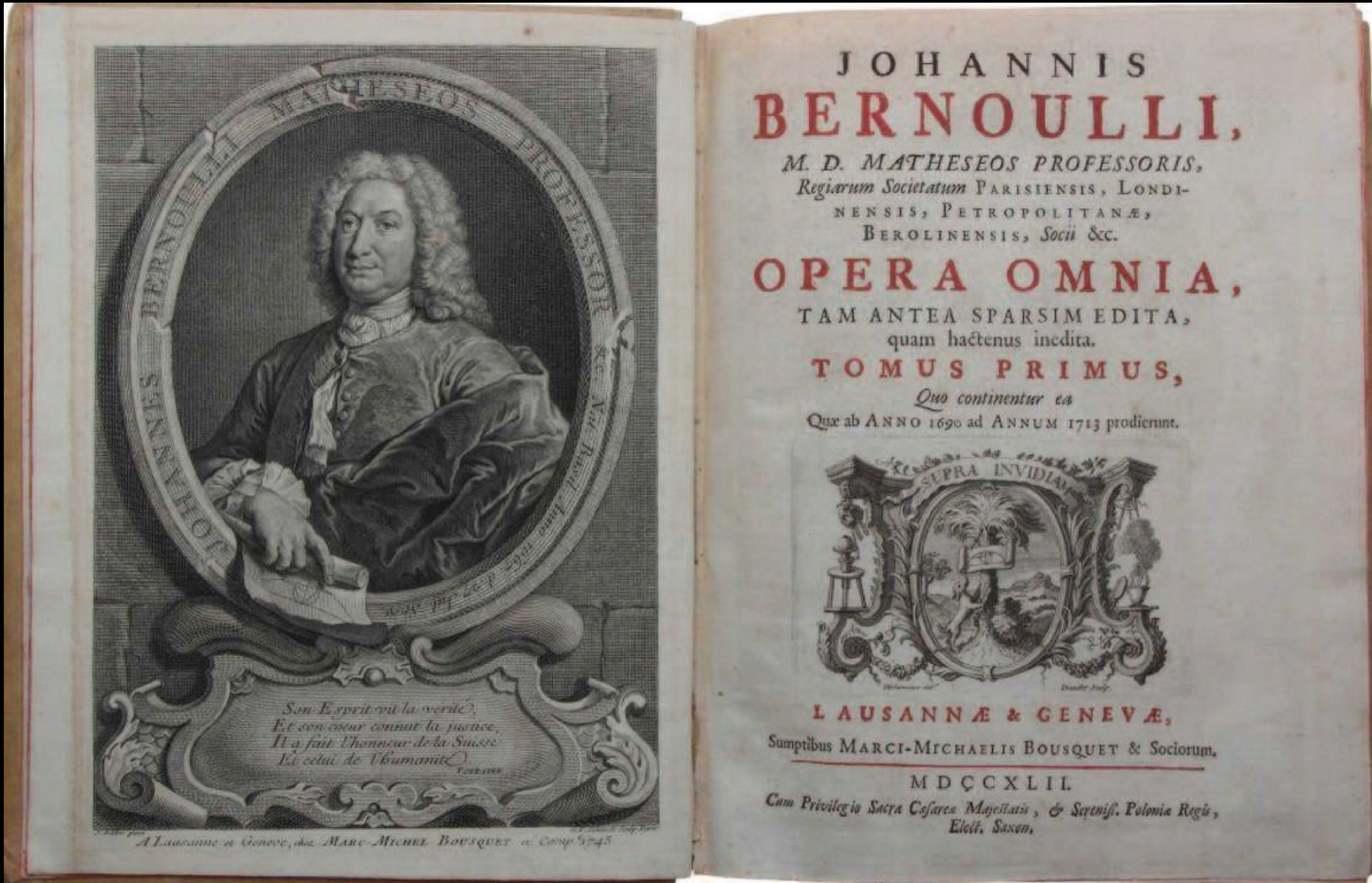
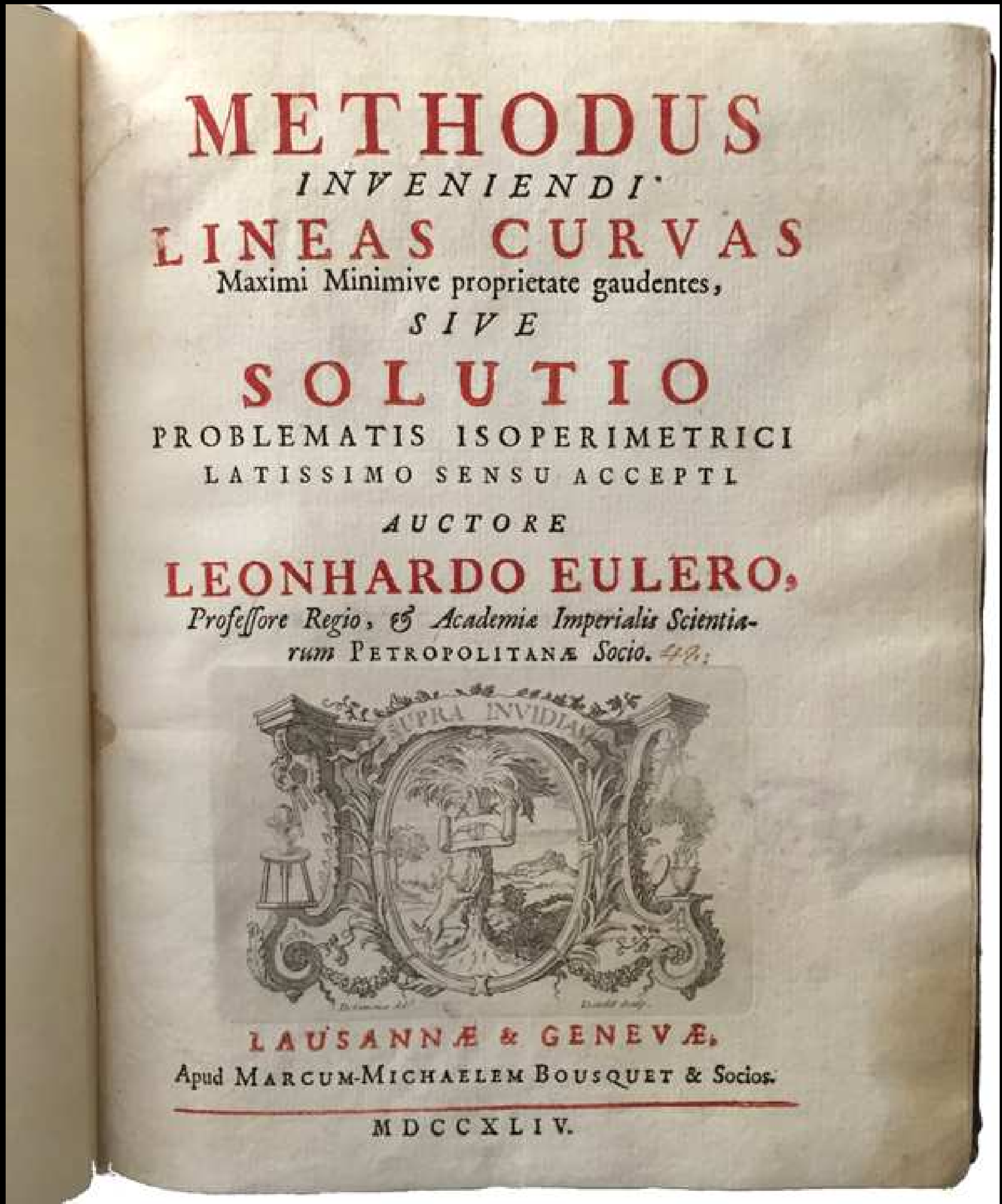
Johann Bernoulli and the Cycloid: A Theorem for Posterity



Frontispiece and title page of Johann Bernoulli's *Opera Omnia* (Bousquet, 1742)

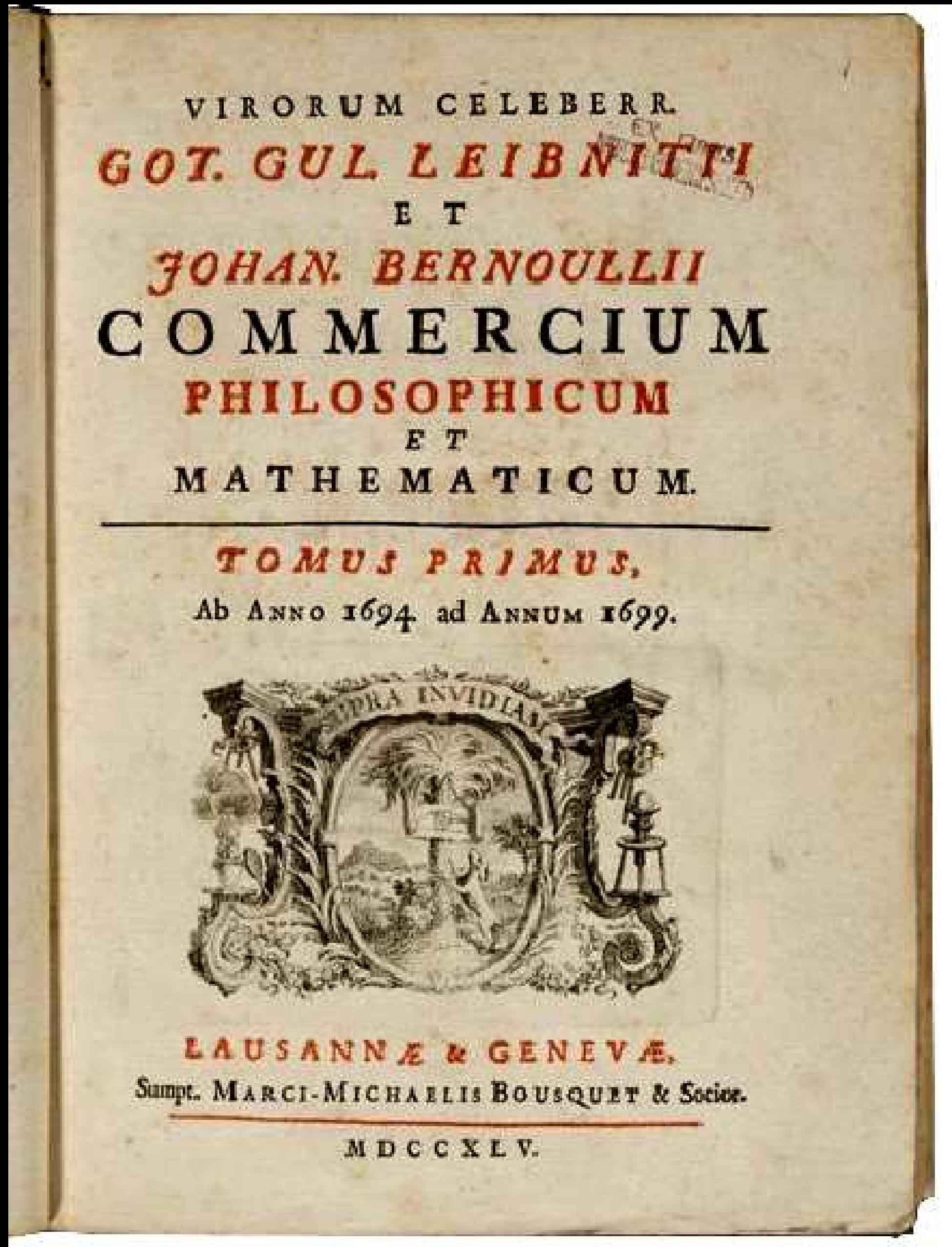
Ph. Henry & G. Wanner, Monte Verità, Sept. 21, 2016



METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

Euler's *Methodus* (Bousquet, 1744)



VIRORUM CELEBERR.
GOT. GUL. LEIBNITII
ET
JOHAN. BERNOULLII
COMMERCIUM
PHILOSOPHICUM
ET
MATHEMATICUM.

TOMUS PRIMUS,
Ab ANNO 1694 ad ANNUM 1699.

LAUSANNÆ & GENEVÆ,
Sumpt. MARCI-MICHAELIS BOUSQUET & Sociorum.
MDCCXLV.

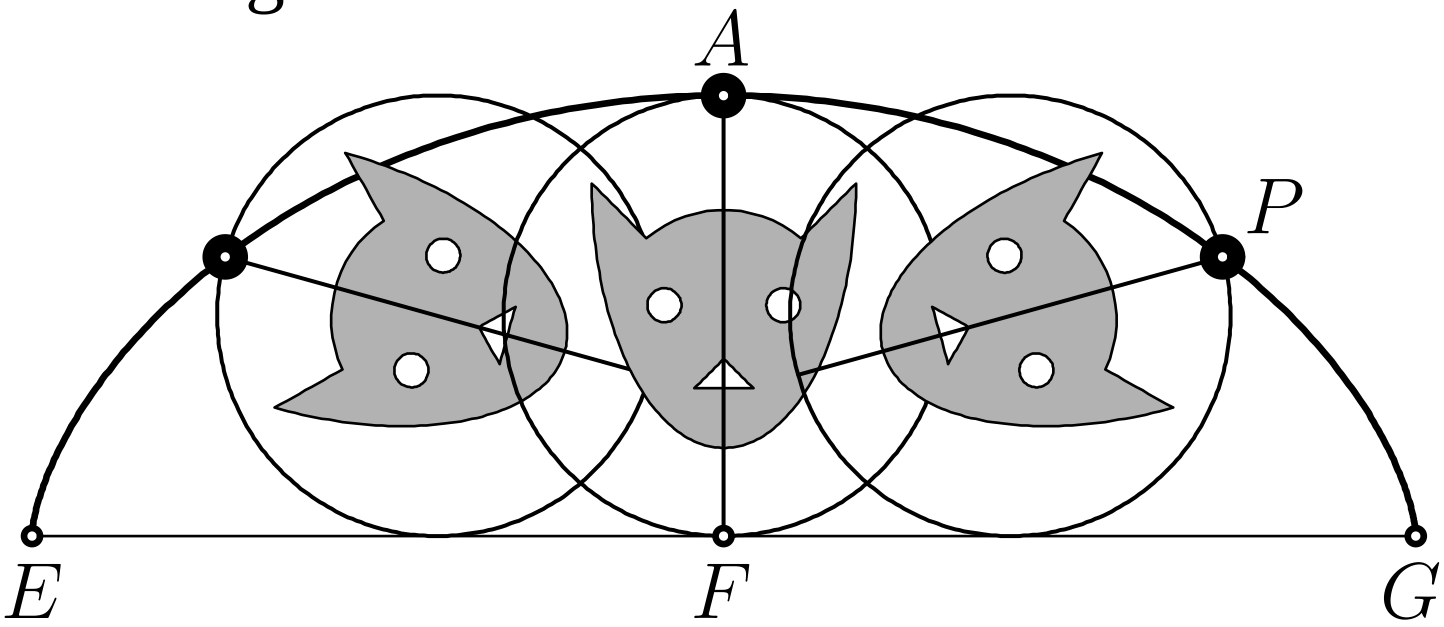
Corr. Leibniz-Bernoulli (Bousquet, 1745)

1. The Beginnings with Geometry

Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola..”

(Galilei, Il Saggiatore, 1623)

Torricelli (1644, *De dimensione parabolæ, Appendix de dimensione cycloidis*, p. 85): Cycloid defined and named 1599 by Galilei as the curve generated by a point P of a “generating circle” AF rolling on a line EFG:...



After long struggle: $\boxed{\text{area}_{\text{cyclo.}} = 3 \cdot \text{area}_{\text{circle}}}$.

B. Pascal (Dec. 1658, *Histoire de la Roulette*):

"Il fit donc imprimer son livre en 1644, dans lequel il attribue à Galilei ce qui est dû au P. Mersenne ... et à soi-même ce qui est dû à M. de Roberval "

"[Pascal] fait preuve d'une extrême partialité vis à vis de Roberval et d'une grande injustice envers Torricelli" (Editeurs des *Œuvres de Pascal*)

Challenge by "Amos Dettonville" (Blaise Pascal, Jun. 1658)

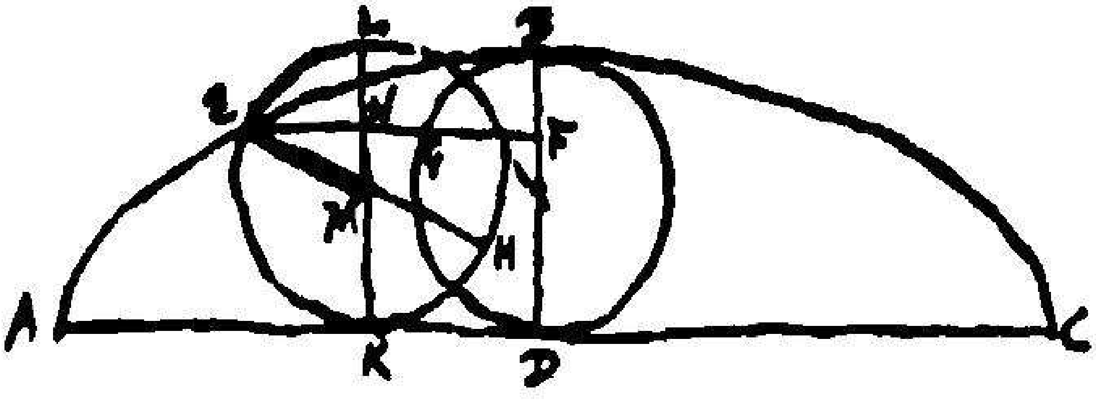
"Ils [Les problèmes proposés par Pascal] me semblent si difficiles pour la pluspart que je doubte fort si celuy mesme qui les a proposez les pourroit tous resoudre, et voudrois bien qu'il nous en eust assuré dans ce mesme imprimé. Autrement il est fort aisé d'inventer des problemes impossibles...."

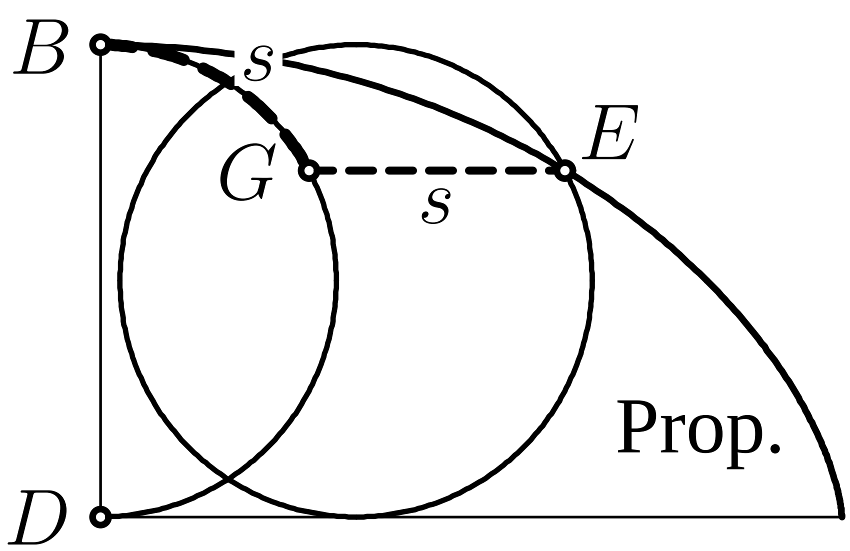
(Letter of Huygens to Ismaël Boulliau, July 25, 1658)

This challenge has boosted researches about this curve again. It is reported that Pascal was searching a terribly difficult geometric problem in order to divert his spirit from painful tooth akes.

Christiaan Huygens (1629–1695). Manuscript (July 1658):

Prop. HM1. arc$BG = GE$.





Proof.

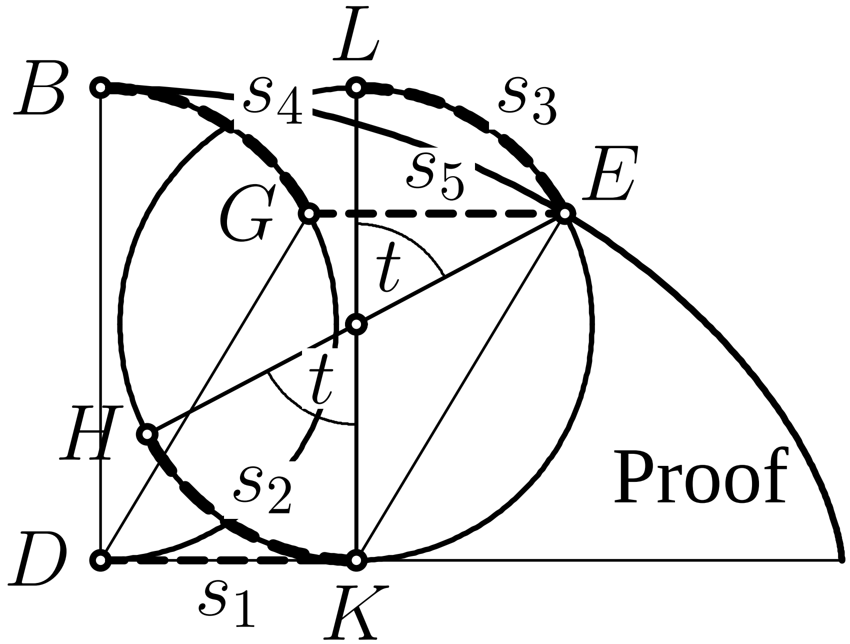
$s_1 = s_2$ (by Def.)

$s_2 = s_3$ (Eucl. I.15)

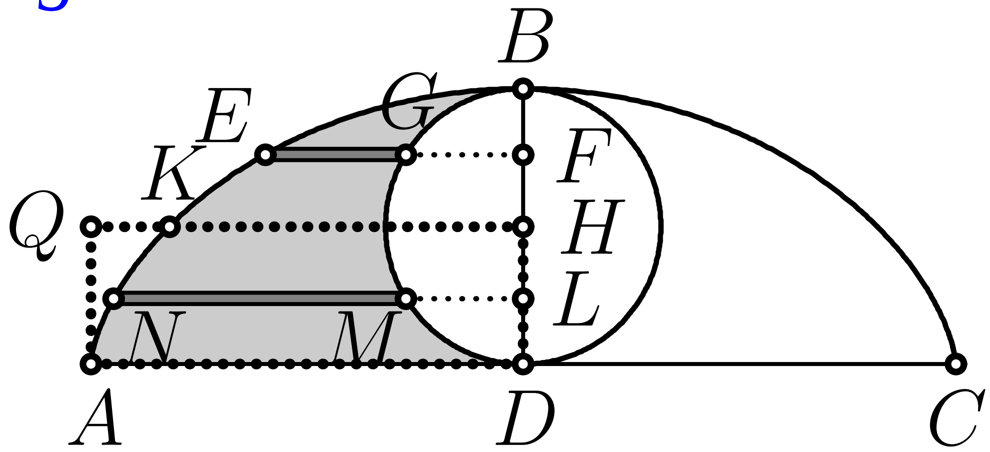
$s_3 = s_4$ (parallel)

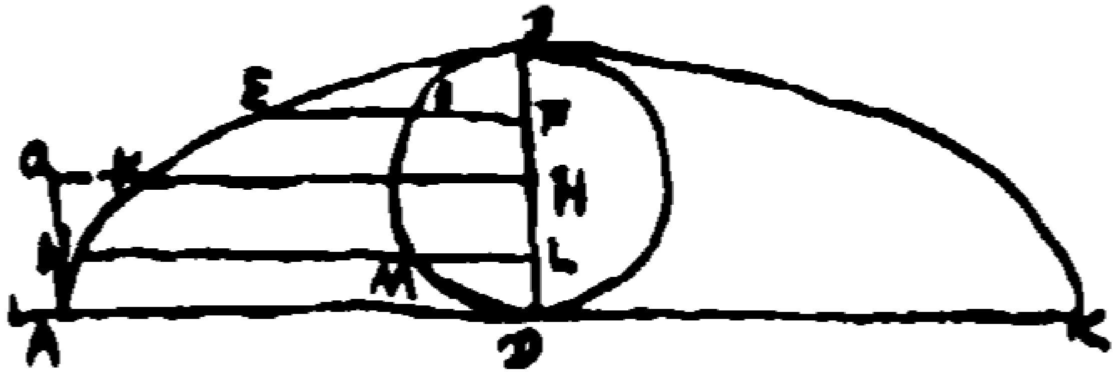
$s_1 = s_5$ (parallelogram)

hence $s_4 = s_5$.



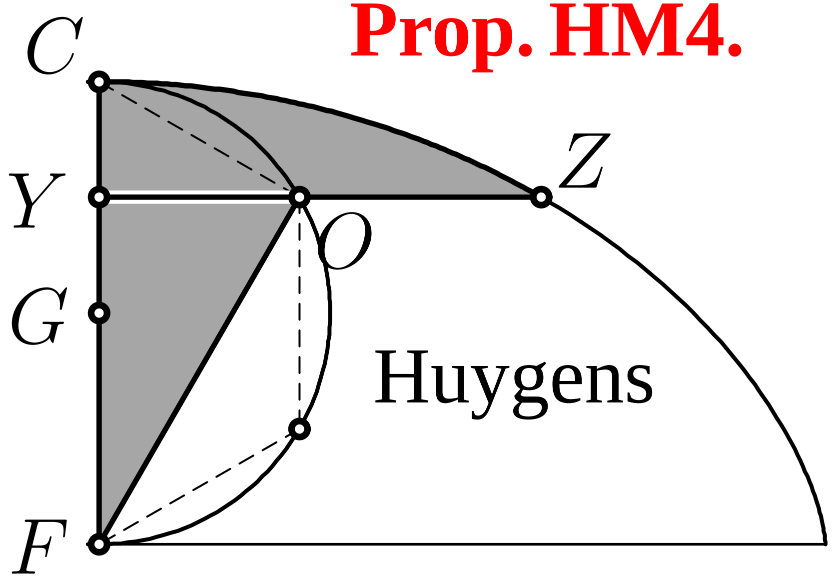
Prop. HM2. *Totum cycloidis spatium ABC triplum erit circuli genitoris BD.*





Proof: $BF = LD \Rightarrow BG = MD \Rightarrow$ (Prop. HM1) $EG + NM = AD$ hinc facile...

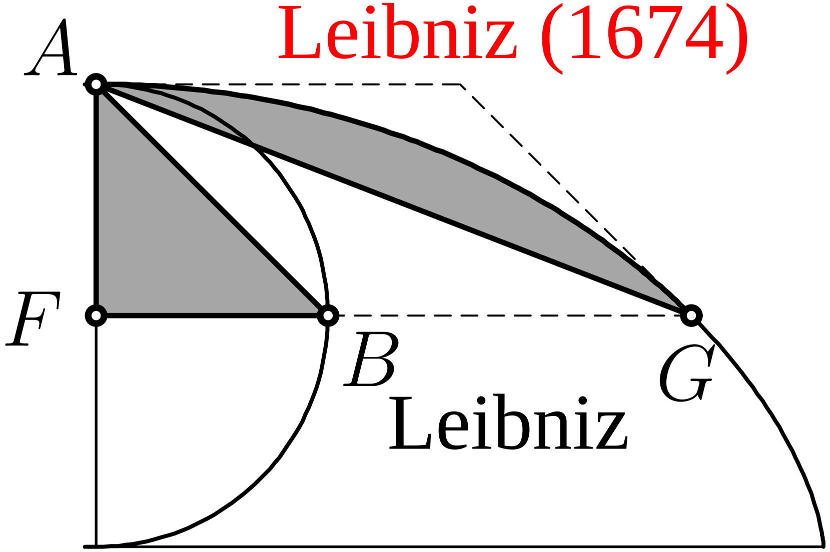
Prop. HM4.



$CY = \frac{1}{4} CF \Rightarrow YZC = FOY$

(long complicated proof)

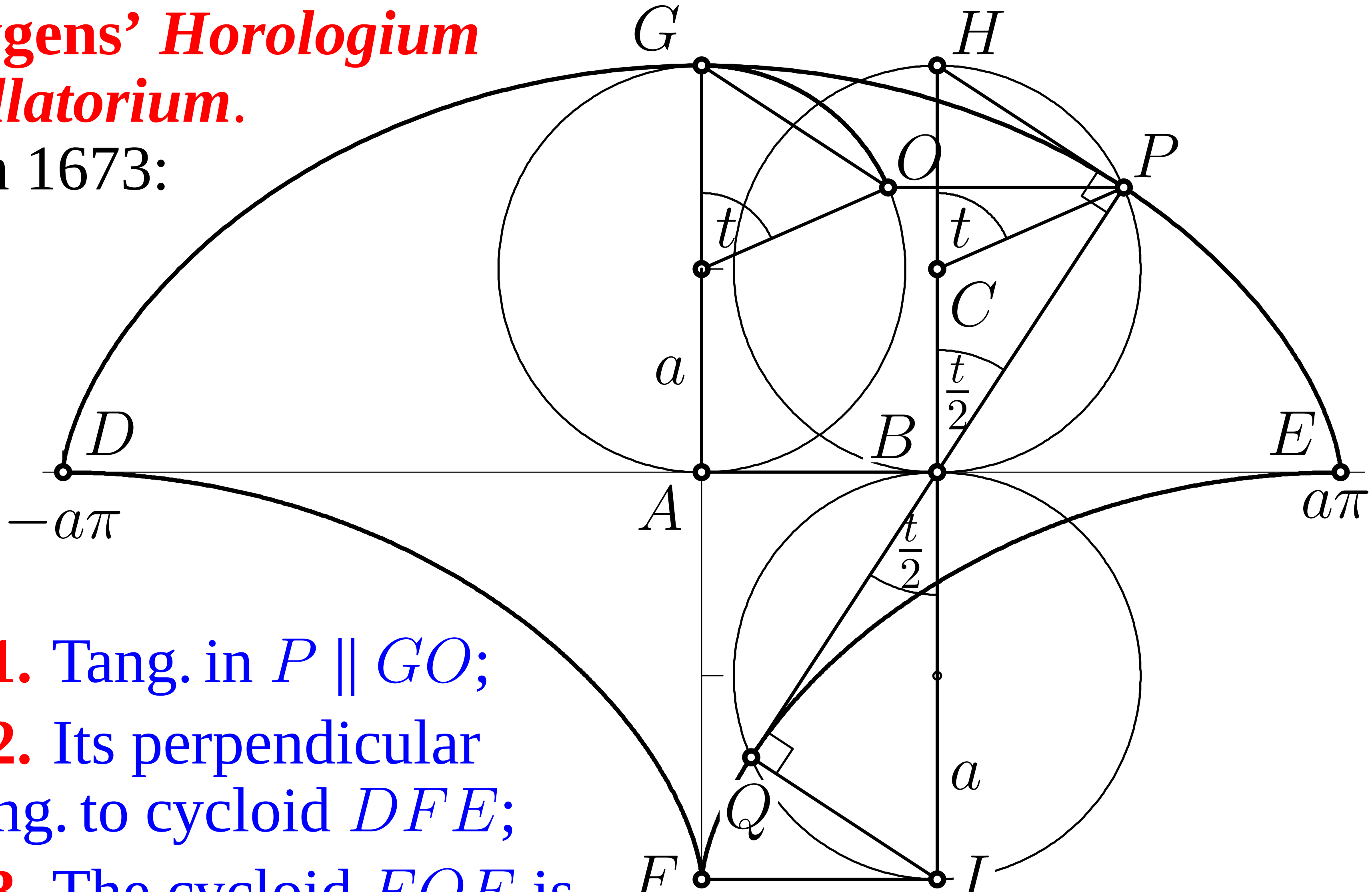
Leibniz (1674)



sector $AG = FBA$

(no proof)

Huygens' *Horologium oscillatorium*.
from 1673:



HH1. Tang. in $P \parallel GO$;

HH2. Its perpendicular is tang. to cycloid DFE;

HH3. The cycloid FQE is evolute of the cycloid GPE, cycloid GPE is involute of FGE.

HH4. The arc length GPE is $4a$, where a is the radius of GA;

HH5. Reversed picture constitutes **isochronous pendulum.**

Proofs fill entire book.

2. New era: Differential- and Integral Calculus

- Newton, *Methodus Fluxionum*; manuscr. 1671, **publ. 1736**
- Leibniz, *Nova methodus*..; AE 1684.

"Es ist aber guth dass wann man etwas wurklich exhibiret man entweder keine demonstration gebe, oder eine solche, dadurch sie uns nicht hinter die schliche kommen" (Leibniz an Bodenhausen, 13./23. März 1691)

Remark: In the same letter Leibniz communicated his solution for Leibniz (1674) (see above) to Bodenhausen.

- Jakob and Johann Bernoulli: years long struggle to decipher Leibniz' "énigme". 1689/90 first successes (see PH and yest.).

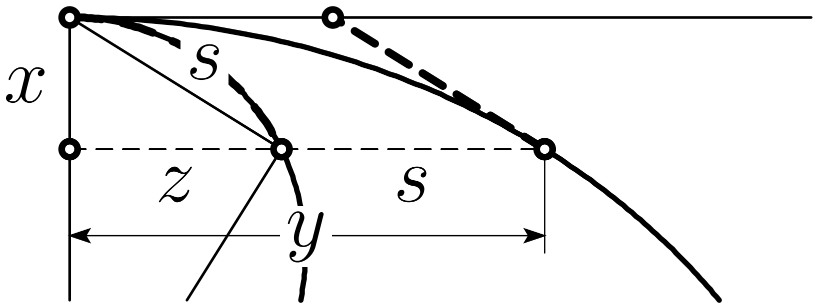
"Ich habe neulich H. Prof. Pfauzio geschrieben daß mich H. Bernoulli Expression nicht wenig geargert, immaßen er darinn nicht wenig gräzgrob, immaßen er darinen nicht allein pretium meines Calculi (dessen er sich doch selbst nun so nützlich bedienet...) zu vermindern suchet, sondern fast mir tacite plagium imputiret..." (Leibniz an Mencken, 19 Mart 1691)

End of 1690: Johann $\Rightarrow$ Geneva, late 1691: $\Rightarrow$ Paris.

"In Genf hatte Johann ... schöne Erfolge erzielt, vor allem fast gleichzeitig mit dem Bruder die Formel $\rho = (\frac{ds}{dx})^3 : \frac{d^2y}{dx^2}$ für den Krümmungshalbmesser aufgestellt und mit ihrer Hilfe grosse formale Vereinfachungen ... erzielt." (J.E. Hofmann 1968; from the preface of reproduction of Johann's *Opera Omnia*, Georg Olms, Hildesheim)

Important source: Lessons for the Marquis de l'Hospital
(published 1742 in vol. III of *Opera Omnia*, p. 385 ff.)

Example: Lectio XVII, proof of **HH1**.



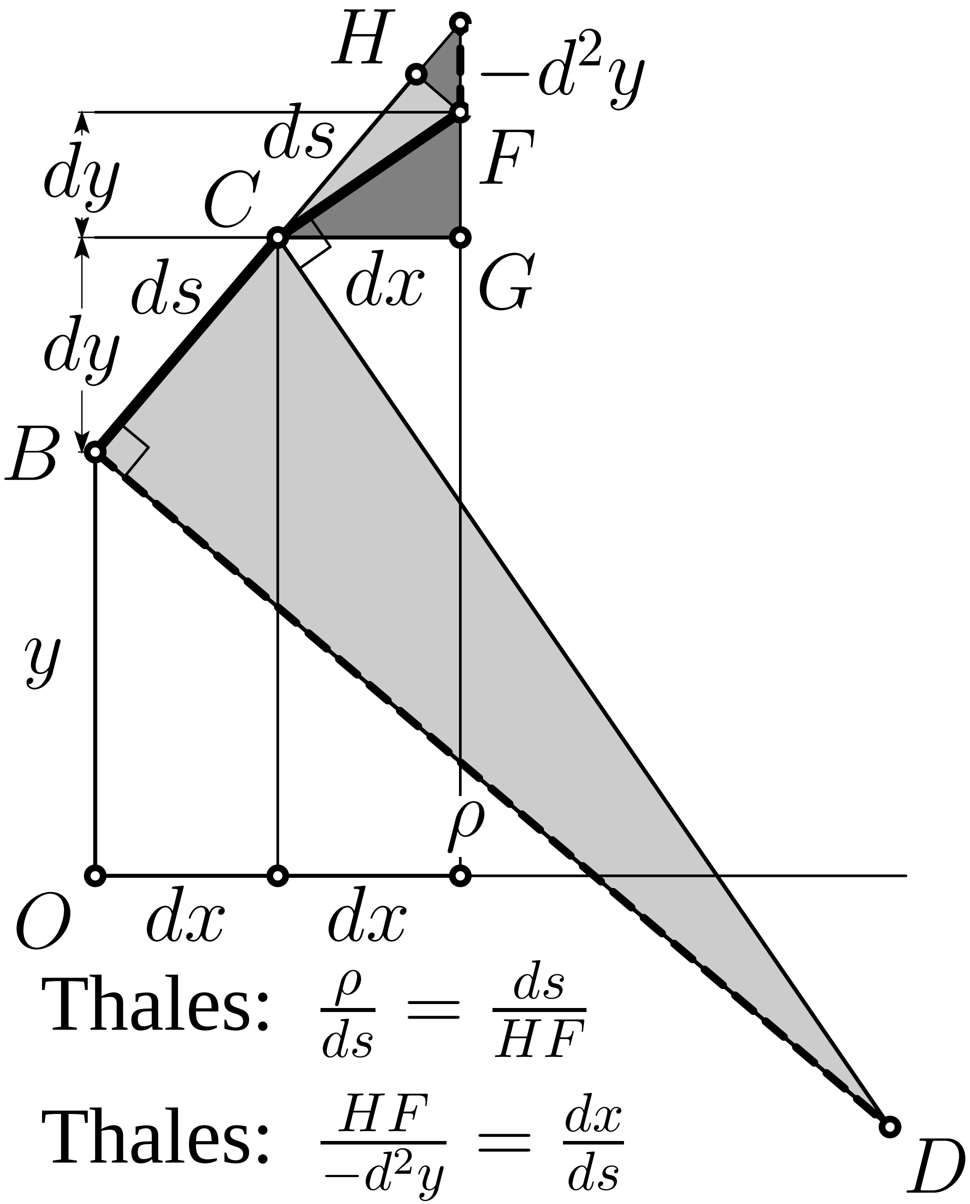
$$z = \sqrt{2ax - x^2}$$
$$ds^2 = dx^2 + dz^2$$

$$\frac{dy}{dx} = \ldots = \frac{z}{x} \quad (*)$$

(see PH, Slide 12-13)

(Hundred year old questions become simple calculations following some simple rules)

Lectio XVI: Circuli Osculantis

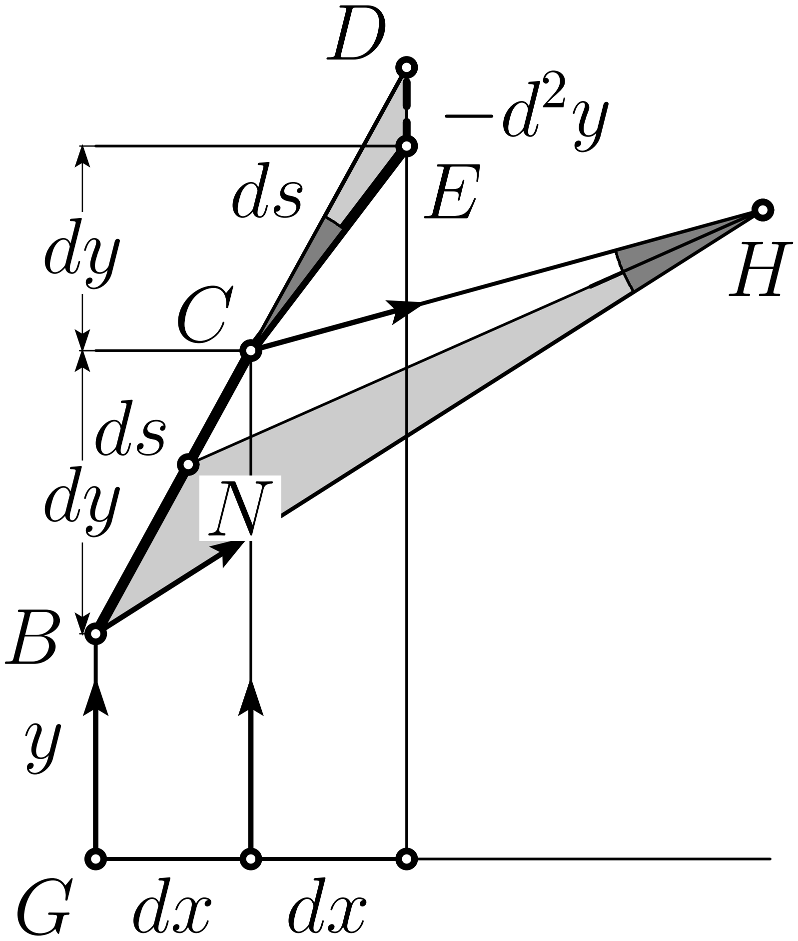


Thales: $\frac{\rho}{ds} = \frac{ds}{HF}$

Thales: $\frac{HF}{-d^2y} = \frac{dx}{ds}$

$$\rho = \frac{(dx^2 + dy^2)\sqrt{dx^2 + dy^2}}{-dx\, d^2y}$$

Lectio XXVI: Curvis Causticis



Thales: $\frac{BH}{BN} = \frac{ds}{-d^2y}$

Eucl. VI.3: $BN \rightarrow \frac{ds}{2}$

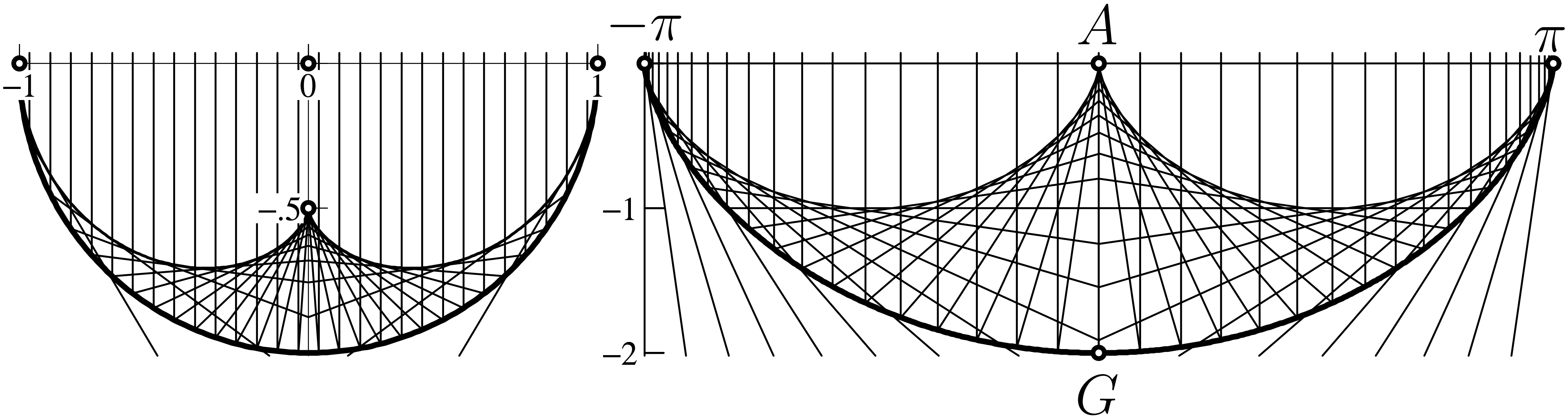
$$\text{BH erit} = \frac{dx^2 + dy^2}{-2\, d^2y}$$

Lectio XXXI.

“Sit Cyclois ABC ...

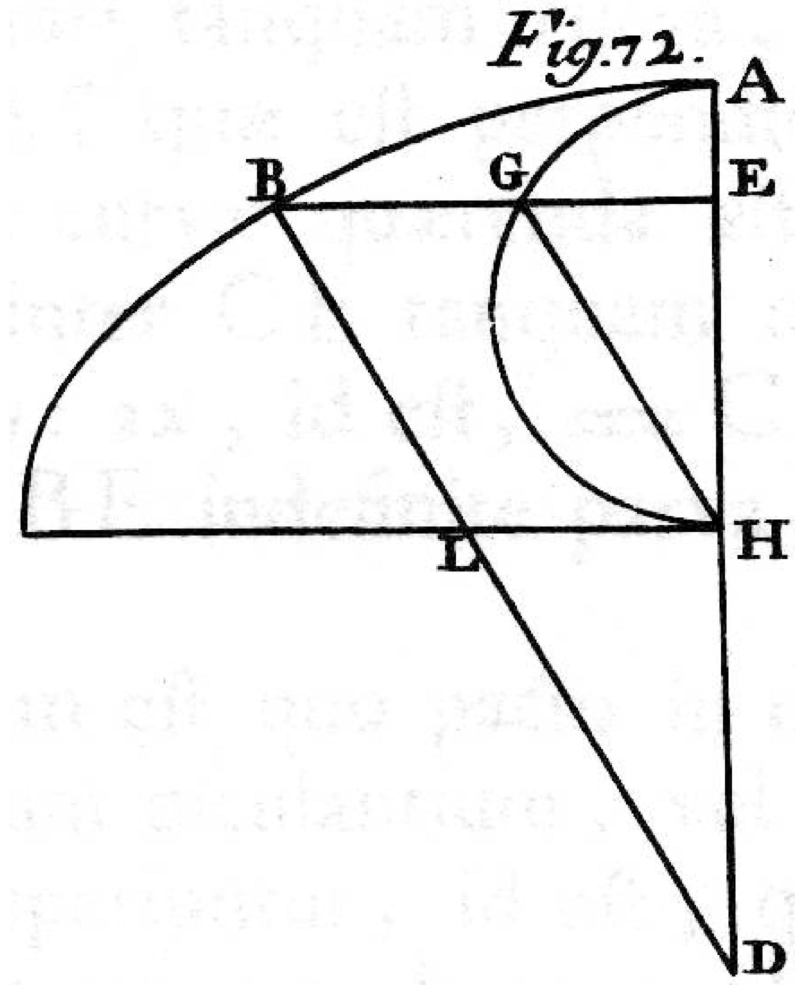
(+ 13 lines of calculations ...)

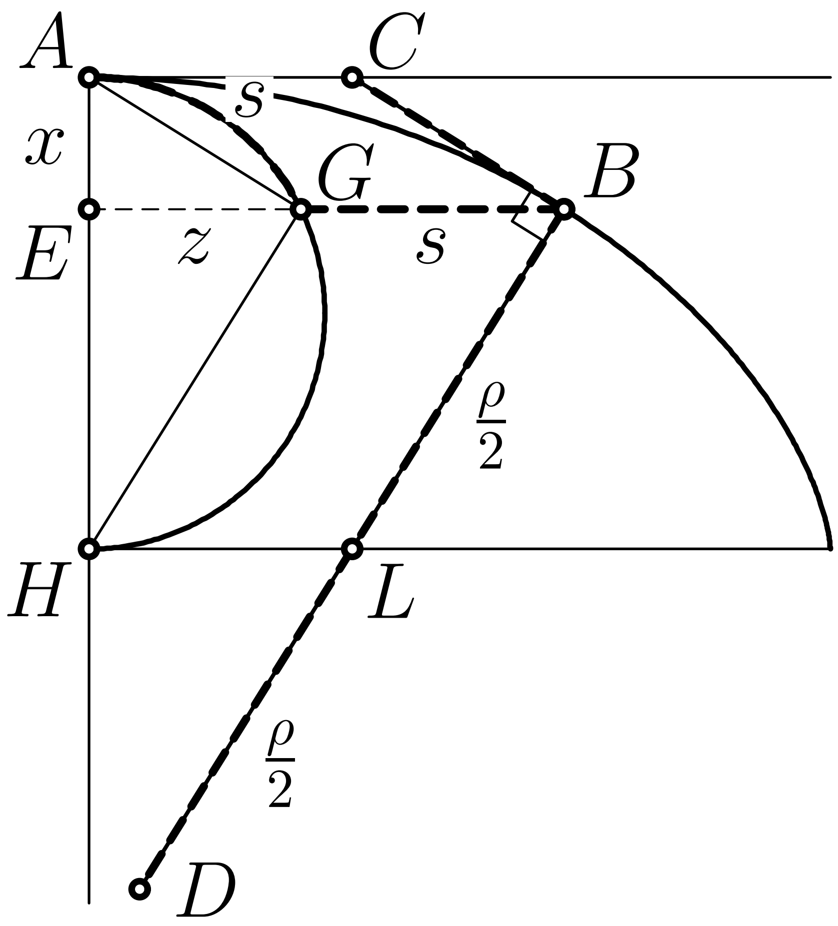
“Caustica AHE est etiam Cyclois, cujus circulus genitor est subquadruplus circuli EFC”



(see talk PH; see also “alterum sic demonstratur” Lectio XXXII, Opera III, p. 479 et Fig. 116. p. 482).

Lectio XVII: “Sit AB Cyclois”.

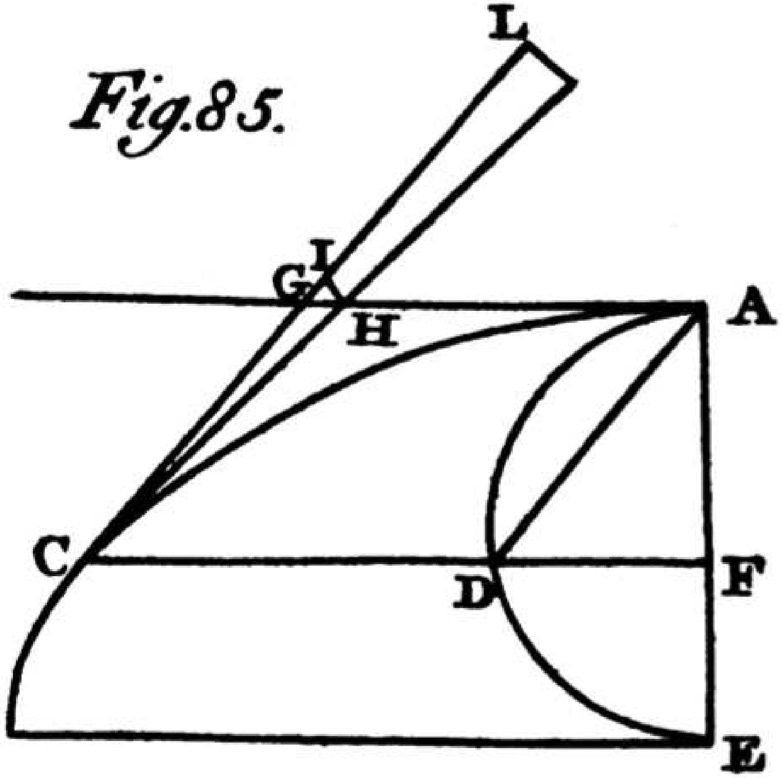


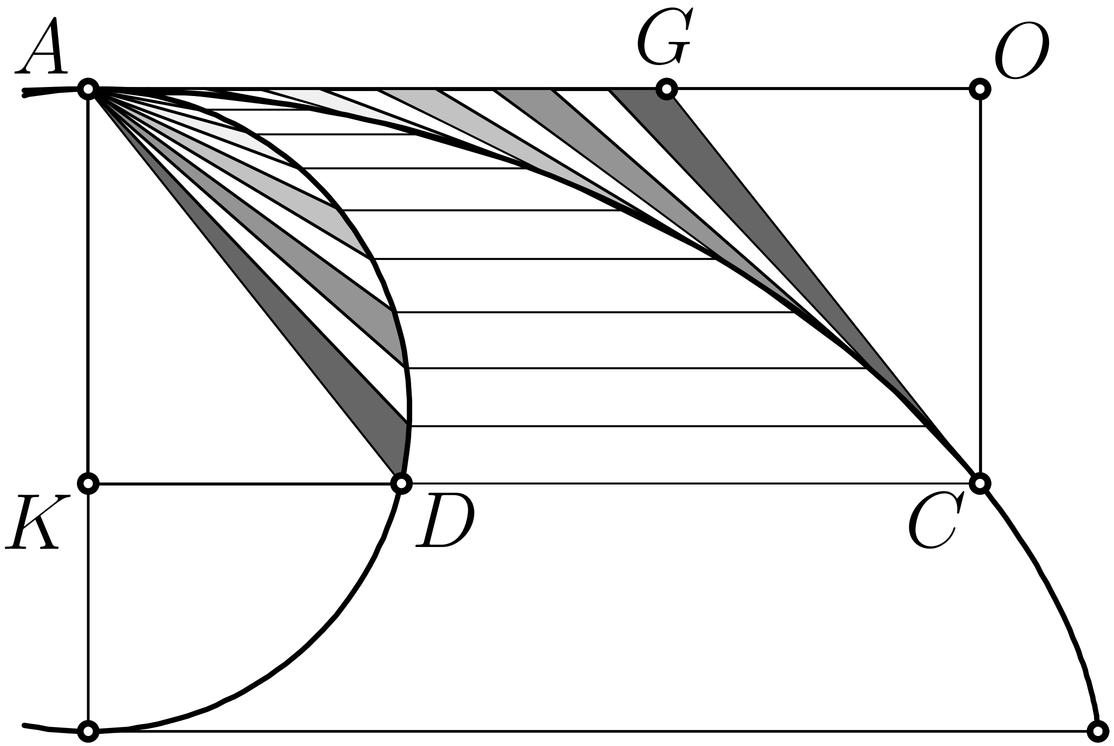


... two more lines of calculations from (*) to obtain $d^2y \Rightarrow$

Lemma 1. ... ideoque $BD = 2\,GH = 2\,BL$.

Lectio XIX: Rectificatione curvarum ope suæ Evolutionis.



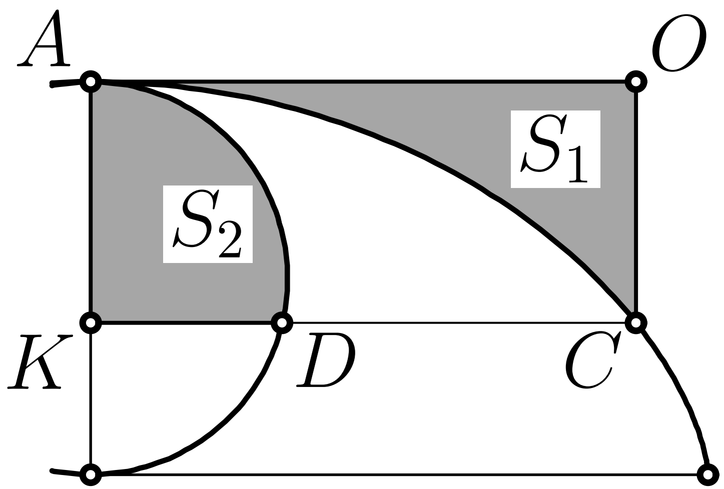


Many parallels HH1: “spatium curvilineum AGC = segm. AD”

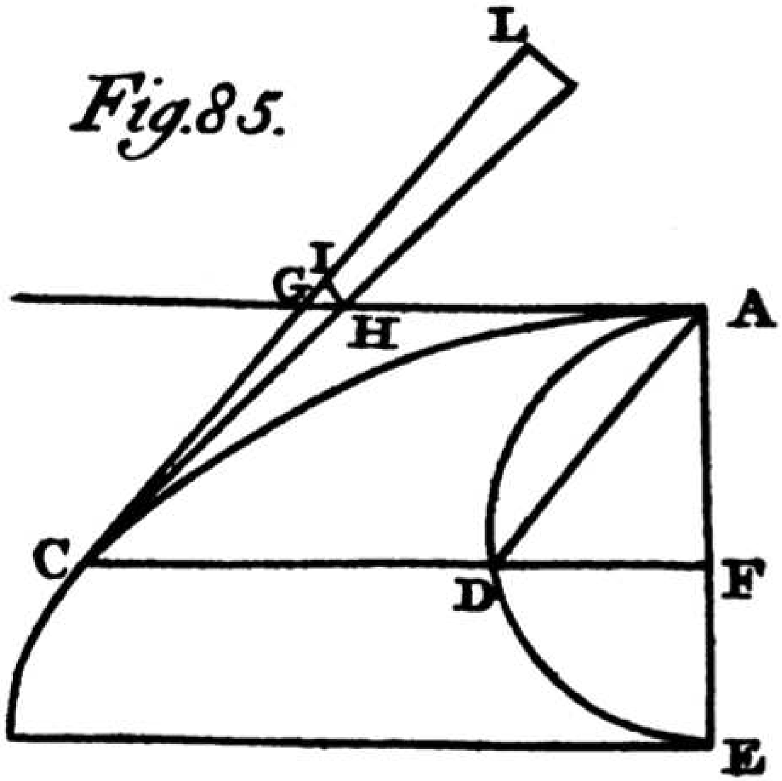
Lemma 2.

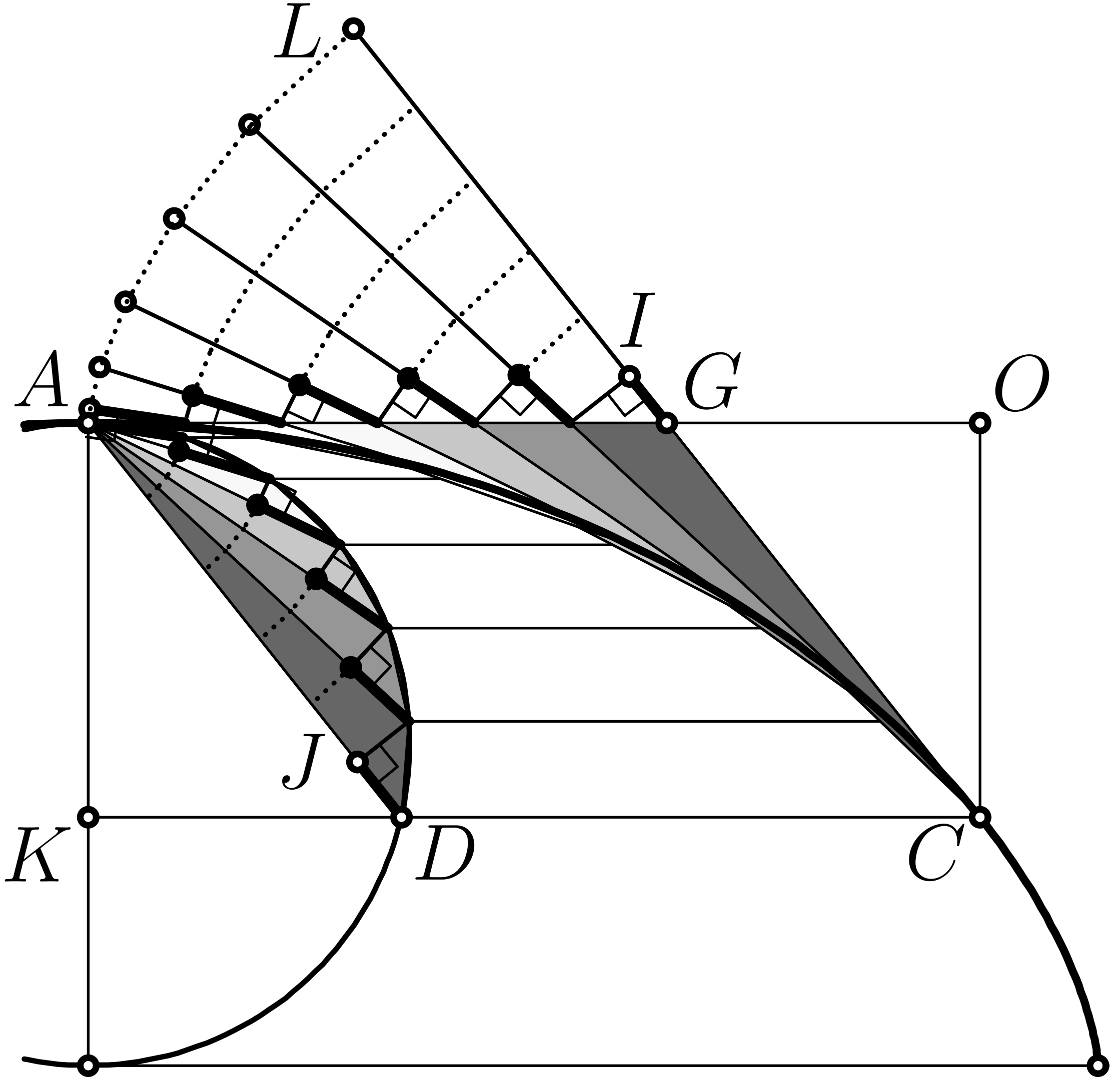
$S_1 = S_2$

(append identical triangles AKD and COG)



Lectio XIX: Continuatio ejusdem argumenti.





$GI = DJ$, "hujus integrale est" $LG = \sqrt{2ax} = AD = CG$

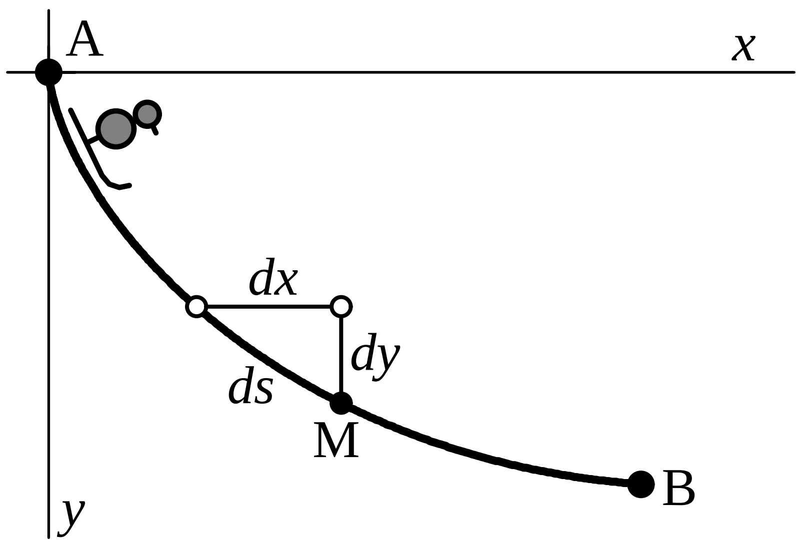
Lemma 3. curvæ $AC =$ duplo $CG = CL$

Inverse of Lemma 1: proofs of **HH2, HH3, HH4** all together.

2.a. The Brachystochrone Problem

"Tous ceux qui sçavent au moins les Nouvelles des Sciences, ont entendu parler du celebre Problême de *la plus vîte Descente*" (Fontenelle, Éloge du Marquis de l'Hospital, p. 51)

At the end of an article in the *Acta Eruditorum* from June 1696, Johann Bernoulli suggests the following problem:



"Datis in plano verticali duobus punctis A et B assignare mobili M, viam AMB per quam gravitate sua descendens et moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B".

He adds that the curve is well known to the geometers and fixes a delay of six months for submitting a solution.

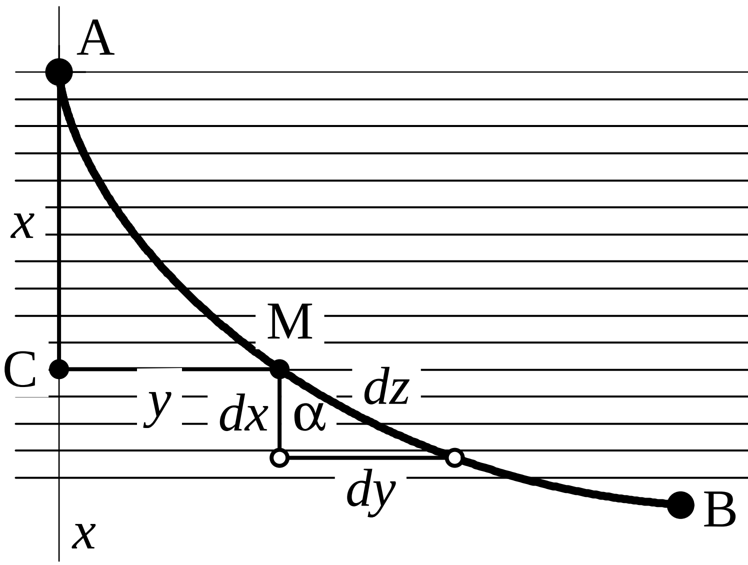
Leibniz (*Journal des Sçavans* du 19 novembre 1696, p. 451-455): delay should be prolonged until Easter 1697, so that also other people might taste the pleasure of the new Calculus.

In the same spirit: “Je voudrois que quelques uns de vos Geometres qui se vantent de posseder de si excellentes methode de *maximis et minimis*, s’y attachassent, car voylà un exemple, qui leur donnera de la besoigne et peutetre plus que leur methode ne pourra faire.” (Joh. Bernoulli, letter to de l’Hospital, June 30, 1696)

The solutions of Johann Bernoulli, Jakob Bernoulli, Leibniz, de l’Hospital, Tschirnhaus and Newton are published in the *Acta Eruditorum* of mai 1697.

Johann's solution.

"une merveilleuse identité de nôtre courbe avec la courbure du rayon de lumière" (Letter of Johann to the Marquis de l'Hospital, march 30, 1697)



Fermat, Leibniz and Huygens:

$$\frac{v_1}{\sin\alpha_1} = \frac{v_2}{\sin\alpha_2}$$

$$\Rightarrow \frac{v}{\sin\alpha} = \frac{v\sqrt{dx^2+dy^2}}{dy} = a$$

Galilei: $v = \sqrt{ax}$

simplify ... and render more complicated ...

$$dy = \sqrt{\frac{x}{a-x}}\,dx = \frac{x\,dx}{\sqrt{ax-x^2}} = \frac{a\,dx}{2\sqrt{ax-x^2}} - \frac{(a-2x)dx}{2\sqrt{ax-x^2}}\,.$$

both terms are now integrable:

repeat: $$dy = \sqrt{\frac{x}{a-x}}\,dx = \frac{\frac{a}{2}\,dx}{\sqrt{ax-x^2}} - \frac{(a-2x)dx}{2\sqrt{ax-x^2}}\,.$$

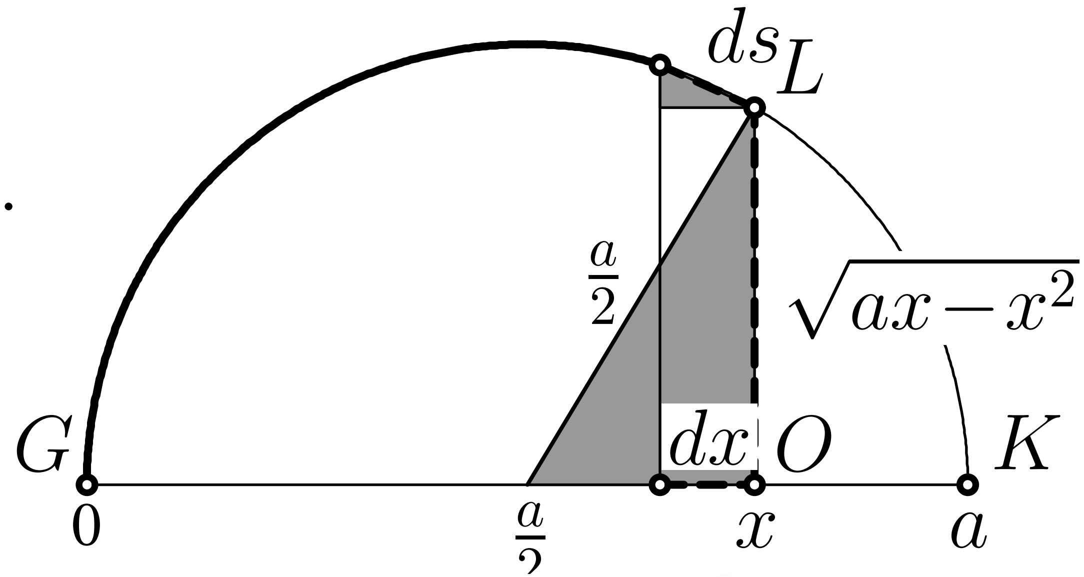
"integralia æquantur"

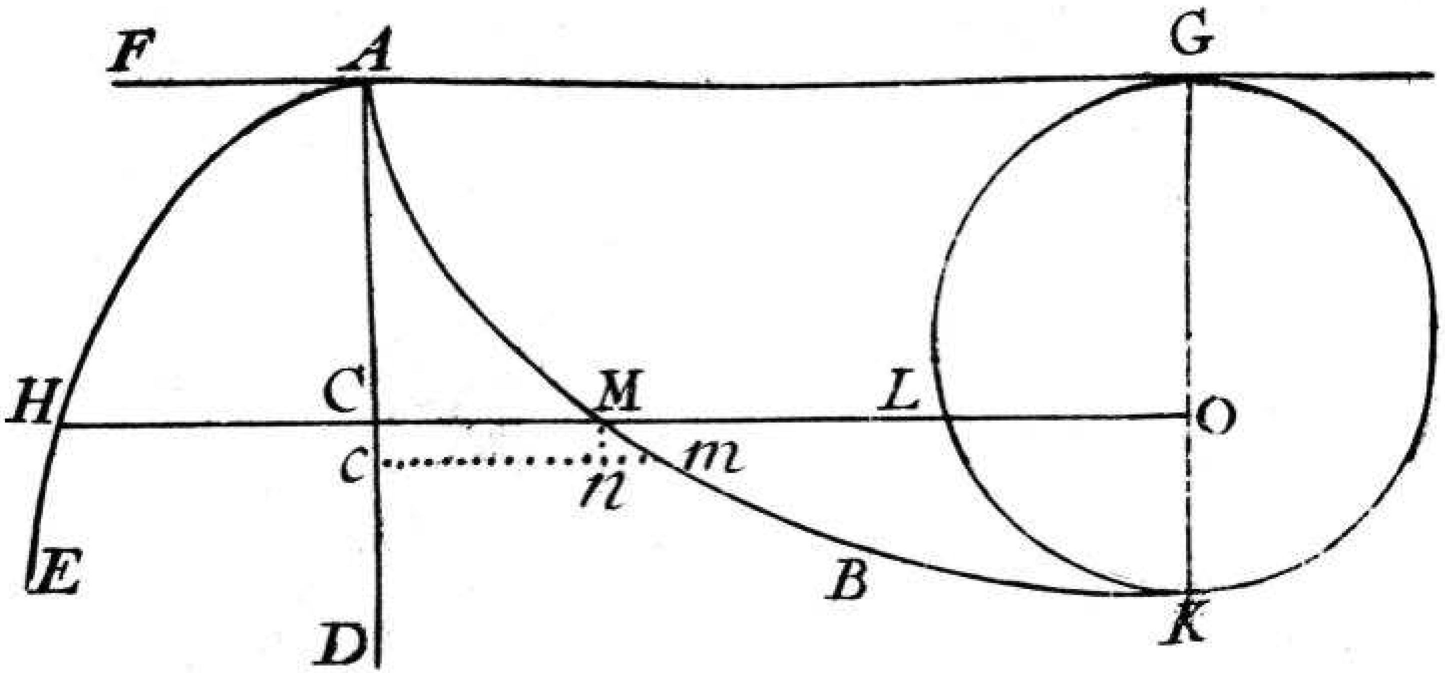
$$y = CM = \text{arc}(GL) - LO\,.$$

circle well positioned:

$$ML = \text{arc}(LK)\,.$$

HM1: sol. is cycloid!

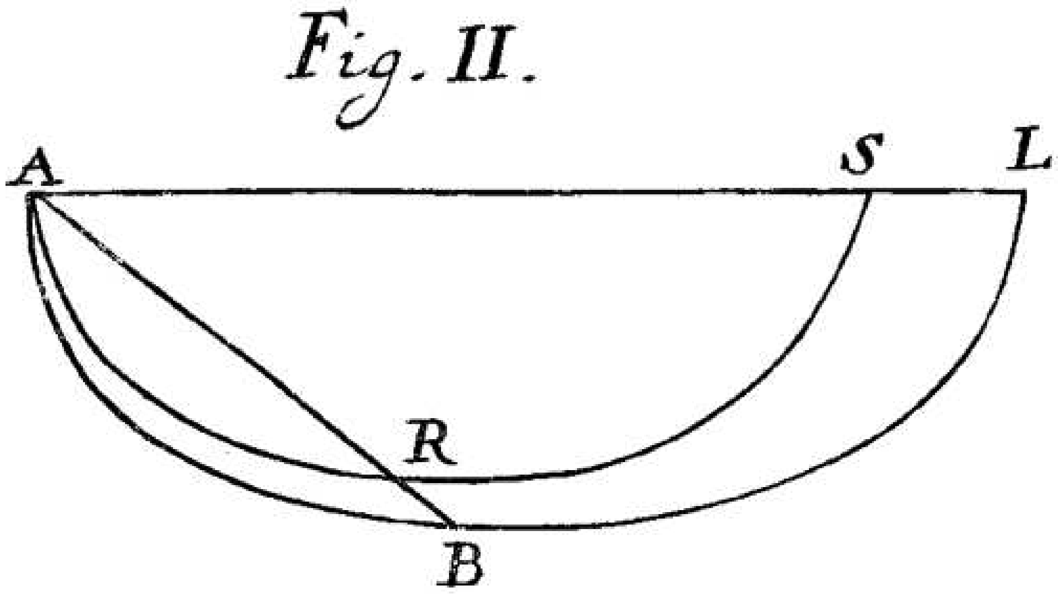




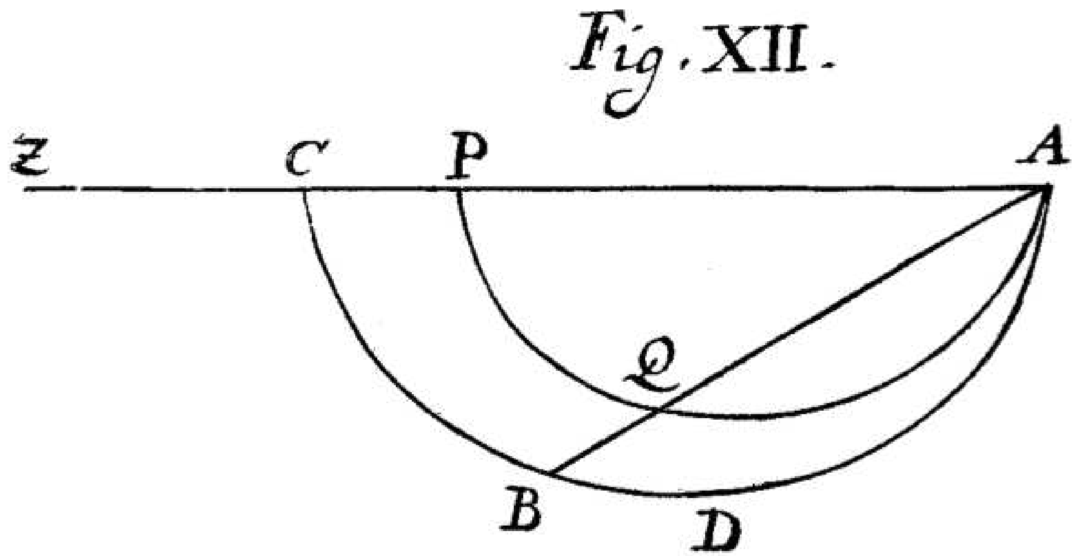
Answer not yet perfect

("Datis in plano verticali duobus punctis A et B etcetera...)!

Use Thales:

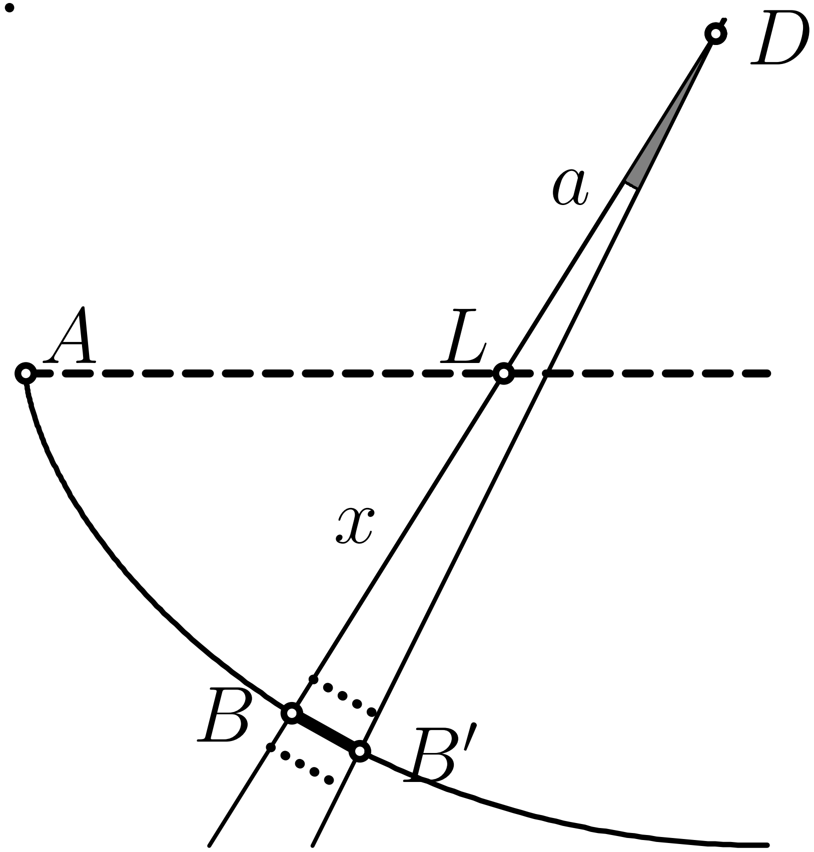


Joh. Bernoulli, AE 1697



Newton AE 1697 **No proof !!**

“*De la plus vîte descente, résolue d’une maniere directe & extraordinaire*”. (communiquée à “L’incomparable Mr. Leibnitz” en 1697, publié AE 1718, *Opera* II, p.267–268).



Let BDB' be fixed sector with fixed distance $LD = a$.

Mobile starting in A arrives in B with veloc. $\sqrt{x}$.

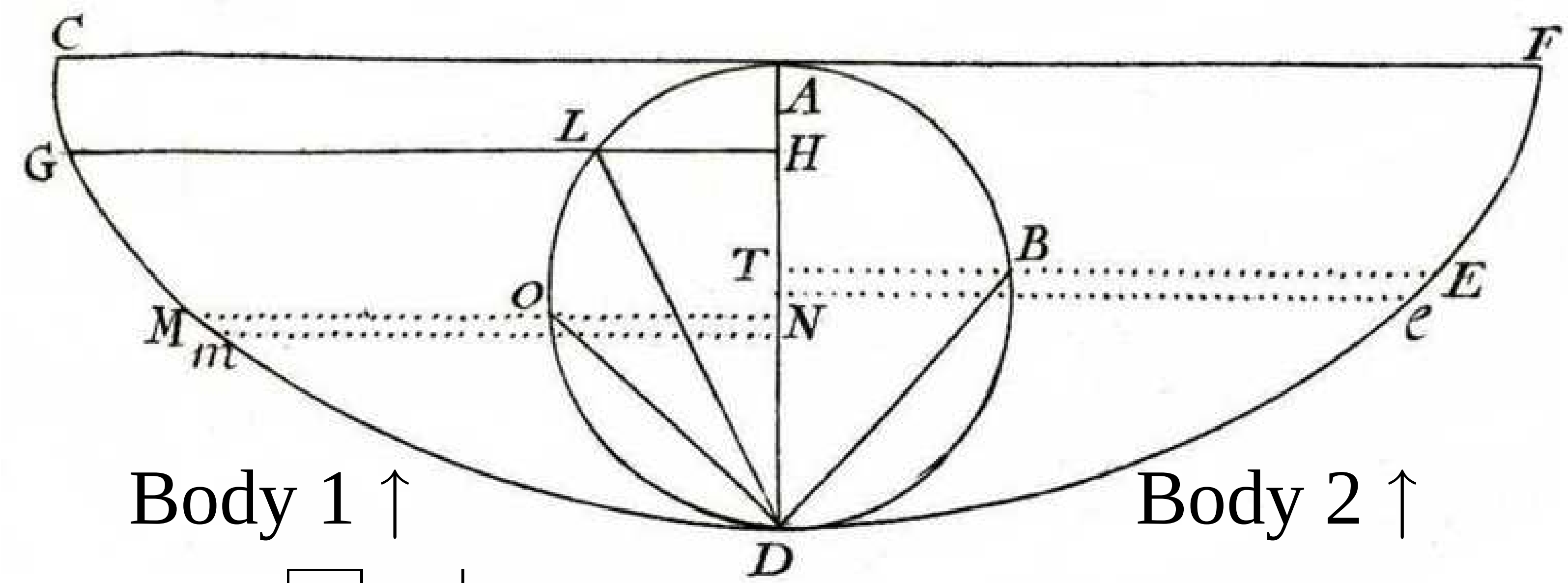
For which x is crossing for sector BB' **in minimal time**?

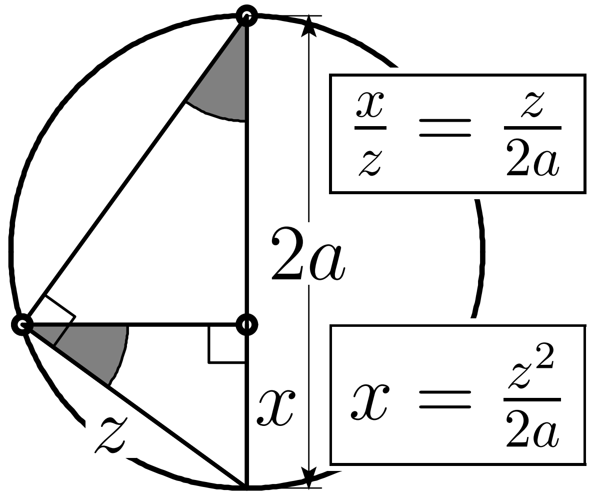
Solution:

$$dT = \frac{ds}{v} = \frac{a+x}{\sqrt{x}} \cdot d\alpha = \text{min!} \quad \Rightarrow \quad \frac{a}{\sqrt{x}} + \sqrt{x} = \text{min!} \quad \Rightarrow \quad \boxed{x = a},$$

i.e. (**Lemma 1**) solution is cycloid.

2.b. Isochronous pendulum. A.E. 1698 (*Opera* I, p. 248):





1	$= q \times$
GD	FD
MD	ED
mD	eD
Mm	Ee

2	$= q \times$
LD	AD
OD	BD

3	$= q^2 \times$
HD	AD
ND	TD
HN	AT

(**1** per Def. **2** "per naturam Cycloidis" (Lemma 3) **3** Thales)

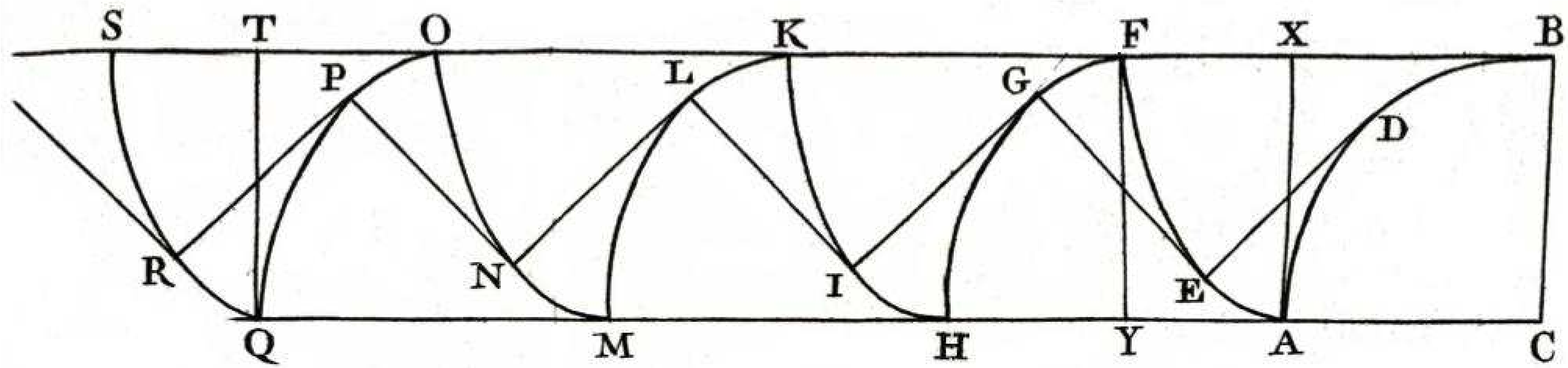
"per naturam gravium descendentium" : $\text{vel.}_{Mm} = q \cdot \text{vel.}_{Ee}$

"descendus per DF & DG sunt isochroni".

2.c. Rectifying the circle through involutes

"Ce théorème remarquable est dû à Jean Bernouilli (...)." (S. D. Poisson, 1820)

Johann Bernoulli gives in vol. IV of his *Opera Omnia* p. 98-108 a surprising method for approximating π.

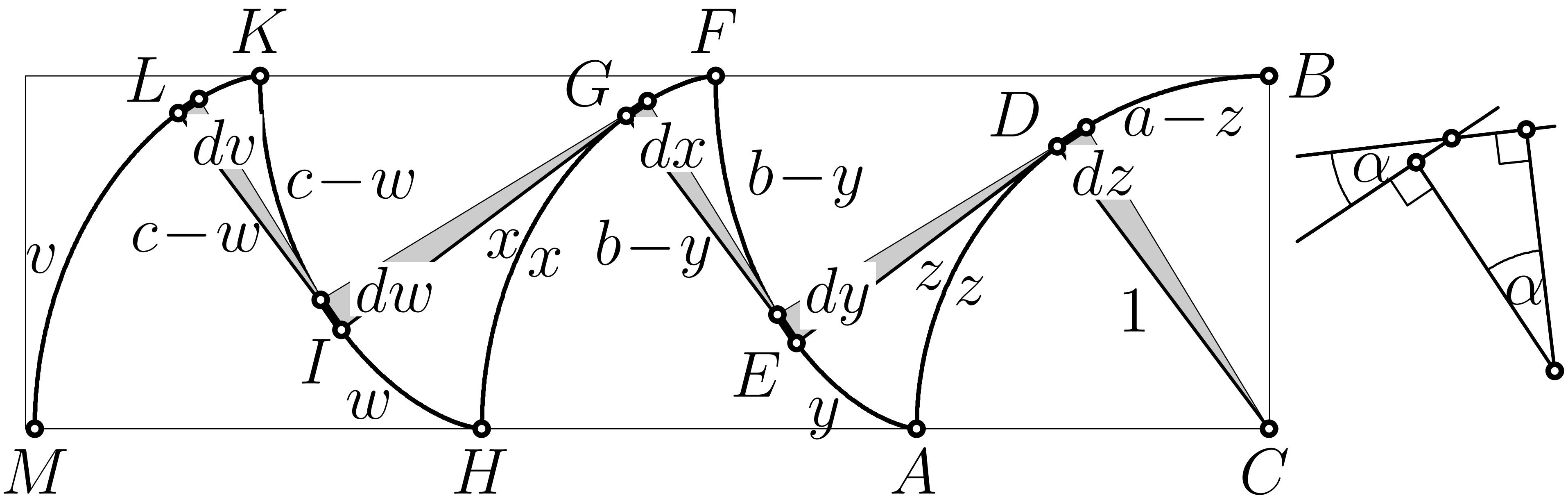


Idea. Start with circle ADB of radius $AC = BC = 1$ and arc length $a =$ arc AB (our $\frac{\pi}{2}$). Draw involutes AEF, FGH, HIK, KLM, &c starting at A, F, H, K, &c. Write

$$b = \text{arc } AF\,, \quad c = \text{arc } HK\,, \quad e = \text{arc } MO\,, \quad \&c$$

for the total arc lengths. Inspired by **HH3** he claims, without proof, that the involutes convergunt ad Cycloidem.

First proof: Euler 1764. Later proofs: Legendre 1817, Poisson 1820 and Puiseux 1844.



Computation of the arc lengths of the involutes:

Choose D on AB with $z =$ arc AD.

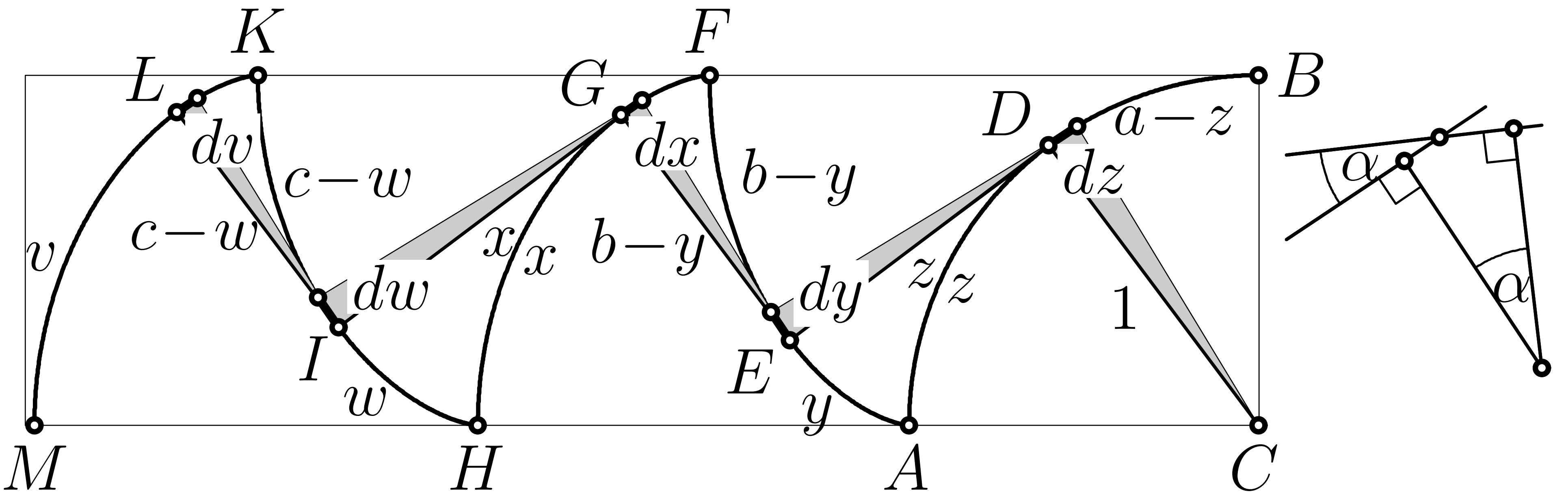
$\Rightarrow$ polygon $D, E, G, I, L, \ldots$ with arc distances $z, y, x, w, v, \ldots$

involutes: tangents of same lengths.

$z = 0$: $y = x = w = v, \ldots = 0$; $z = a$: $y = b$, $w = c$, &c.

Orthogonal angles: all shaded triangles similar, hence

$\frac{dz}{1} = \frac{dy}{z}$, $\frac{dz}{1} = \frac{dx}{b-y}$, $\frac{dz}{1} = \frac{dw}{x}$, etc.



differentials	integration	set $z = a$
$dy = z\,dz$	$y = \frac{z^2}{2!}$	$b = \frac{a^2}{2!}$
$dx = (b - y)\,dz$	$x = bz - \frac{z^3}{3!}$	$\mathit{HF} = ba - \frac{a^3}{3!}$
$dw = x\,dz$	$w = b\frac{z^2}{2!} - \frac{z^4}{4!}$	$c = b\frac{a^2}{2!} - \frac{a^4}{4!}$
$dv = (c - w)\,dz$	$v = cz - b\frac{z^3}{3!} + \frac{z^5}{5!}$	$\mathit{MK} = ca - b\frac{a^3}{3!} + \frac{a^5}{5!}$
$du = v\,dz$	$u = c\frac{z^2}{2!} - b\frac{z^4}{4!} + \frac{z^6}{6!}$	$e = c\frac{a^2}{2!} - b\frac{a^4}{4!} + \frac{a^6}{6!}$

$$\text{Curva I} = a$$

$$\ldots\ldots \text{II} = b = \frac{aa}{1.2}$$

$$\ldots\ldots \text{III} = ba - \frac{a^3}{1.2.3}$$

$$\ldots\ldots \text{IV} = c = \frac{baa}{1.2} - \frac{a^4}{1.2.3.4}$$

$$\ldots\ldots \text{V} = ca - \frac{ba^3}{1.2.3} + \frac{a^5}{1.2.3.4.5}$$

$$\ldots\ldots \text{VI} = e = \frac{caa}{1.2} - \frac{ba^4}{1.2.3.4} + \frac{a^6}{1.2.3\ldots6}$$

$$\ldots\ldots \text{VII} = ea - \frac{ca^3}{1.2.3} + \frac{ba^5}{1.2.3.4.5} - \frac{a^7}{1.2.3..7}$$

$$\ldots \text{VIII} = f - \frac{eaa}{1.2} - \frac{ca^4}{1.2.3.4} + \frac{ba^6}{1.2.3..6} - \frac{a^8}{1.2.3\ldots8}$$

$$\ldots \text{IX} = fa = \frac{ea^3}{1.2.3} + \frac{ca^5}{1.2.3.4.5} - \frac{ba^7}{1.2.3\ldots7} + \frac{a^9}{1.2.3\ldots9}$$

$$\text{X} = g = \frac{faa}{1.2} - \frac{ea^4}{1.2.3.4} + \frac{ca^6}{1.2.3\ldots6} - \frac{ba^8}{1.2.3\ldots8} + \frac{a^{10}}{1.2.3\ldots10}$$

$$\text{XI} = ga - \frac{fa^3}{1.2.3} + \frac{ea^5}{1.2.3.4.5} - \frac{ca^7}{1.2.3\ldots.7} + \frac{ba^9}{1.2.3\ldots9} - \frac{a^{11}}{1.2.3\ldots11}$$

$$\text{XII} = h = \frac{gaa}{1.2} - \frac{fa^4}{1.2.3.4} + \frac{ea^6}{1.2.3\ldots6} - \frac{ca^8}{1.2.3\ldots8} + \frac{ba^{10}}{1.2.3\ldots10} - \frac{a^{12}}{1.2.3\ldots12}$$

Inserting, finally, the computed values of b, c, &c in the subsequent expressions, Johann obtains $HF = 2\frac{a^3}{3!}$, $c = 5\frac{a^4}{4!}$, $MK = 16\frac{a^5}{5!}$, $e = 61\frac{a^6}{6!}$ and so on. See his values :

Curva I $= a\,(1)$	VIII $= a^8\left(\frac{1385}{1.2.3\ldots\ldots 8}\right)$
.... II $= a^2\left(\frac{1}{1.2}\right)$	. IX $= a^9\left(\frac{7936}{1.2.3\ldots\ldots 9}\right)$
.... III $= a^3\left(\frac{2}{1.2.3}\right)$	. X $= a^{10}\left(\frac{50521}{1.2.3\ldots\ldots 10}\right)$
.... IV $= a^4\left(\frac{5}{1.2.3.4}\right)$	. XI $= a^{11}\left(\frac{353792}{1.2.3\ldots\ldots 11}\right)$
.... V $= a^5\left(\frac{16}{1.2.3.4.5}\right)$	XII $= a^{12}\left(\frac{2702765}{1.2.3\ldots\ldots 12}\right)$
.... VI $= a^6\left(\frac{61}{1.2.3\ldots 6}\right)$	XIII $= a^{13}\left(\frac{22368256}{1.2.3\ldots\ldots 13}\right)$
.... VII $= a^7\left(\frac{272}{1.2.3\ldots 7}\right)$	XIV $= a^{14}\left(\frac{199360981}{1.2.3\ldots\ldots 14}\right)$

Curiouso $1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, \ldots$?

Numeri di permutazioni alternanti !! (vede conf. Delucchi.)

I pari sono i *numeri di Euler*:

$1, -1, 5, -61, 1385, -50521, 2702765, -199360981, 19391512145,$

Indeed, the formulas $b - \frac{a^2}{2!} = 0$, $c - b\frac{a^2}{2!} + \frac{a^4}{4!} = 0$, etc. from the rightmost column tell us that the formal series expansion of

$$(1 + b\zeta^2 + c\zeta^4 + e\zeta^6 + \ldots) \cdot (1 - \frac{a^2}{2!}\zeta^2 + \frac{a^4}{4!}\zeta^4 - \frac{a^6}{6!}\zeta^6 + \ldots) = 1$$

is satisfied. Therefore $b, c, e, \ldots$ are the Taylor coefficients of the function $\frac{1}{\cos a\zeta}$. Similarly, the arc lengths for odd roman numbers are the Taylor coefficients of $\tan a\zeta$.

Rectifying the circle. Johann presents two methods:

First method. Limiting cycloid stretch over $CB = AX = YF = QT = 1$, hence generating radius is $r = \frac{1}{\pi}$; thus by **HH4** the arc lengths become $\frac{4}{\pi}$. This we compare with the computed values:

for example

for "Curva VIII"

$\frac{4}{\pi} \simeq \left(\frac{\pi}{2}\right)^8 \frac{1385}{8!}$

hence

$\left(\frac{\pi}{2}\right)^9 \simeq 2 \cdot \frac{8!}{1385}$

or

$\pi \simeq 2\sqrt[9]{\frac{2\cdot 8!}{1385}}$

"Curva"	$2a \simeq \pi$	error
I	$2\sqrt{\frac{2\cdot 1!}{1}}$	0.3131655288
II	$2\sqrt[3]{\frac{2\cdot 2!}{1}}$	-0.0332094503
III	$2\sqrt[4]{\frac{2\cdot 3!}{2}}$	0.0114234934
IV	$2\sqrt[5]{\frac{2\cdot 4!}{5}}$	-0.0024196887
V	$2\sqrt[6]{\frac{2\cdot 5!}{16}}$	0.0007570486
VI	$2\sqrt[7]{\frac{2\cdot 6!}{61}}$	-0.0001999873
VII	$2\sqrt[8]{\frac{2\cdot 7!}{272}}$	0.0000609333
VIII	$2\sqrt[9]{\frac{2\cdot 8!}{1385}}$	-0.0000175640
IX	$2\sqrt[10]{\frac{2\cdot 9!}{7936}}$	0.0000053536

Second method:

divide consecutive results; get no roots but is less precise.

Example: divide VII and VIII :

$$\frac{272a^7}{7!} \simeq \frac{1385a^8}{8!}$$

hence

$$2a \simeq 2 \cdot \frac{8 \cdot 272}{1385} = \frac{4352}{1385}$$

	$2a$	error
I = II	$2 \cdot \frac{2 \cdot 1}{1}$	−0.85840734
II = III	$2 \cdot \frac{3 \cdot 1}{2}$	0.14159265
III = IV	$2 \cdot \frac{4 \cdot 2}{5}$	−0.05840734
IV = V	$2 \cdot \frac{5 \cdot 5}{16}$	0.01659265
V = VI	$2 \cdot \frac{6 \cdot 16}{61}$	−0.00594833
VI = VII	$2 \cdot \frac{7 \cdot 61}{272}$	0.00188677
VII = VIII	$2 \cdot \frac{8 \cdot 272}{1385}$	−0.00064561
VIII = IX	$2 \cdot \frac{9 \cdot 1385}{7936}$	0.00021160
IX = X	$2 \cdot \frac{10 \cdot 7936}{50521}$	−0.00007120
X = XI	$2 \cdot \frac{11 \cdot 50521}{353792}$	0.00002359
XI = XII	$2 \cdot \frac{12 \cdot 353792}{2702765}$	−0.00000789
XII = XIII	$2 \cdot \frac{13 \cdot 2702765}{22368256}$	0.00000262

2.d. Squarable areas of the cycloid

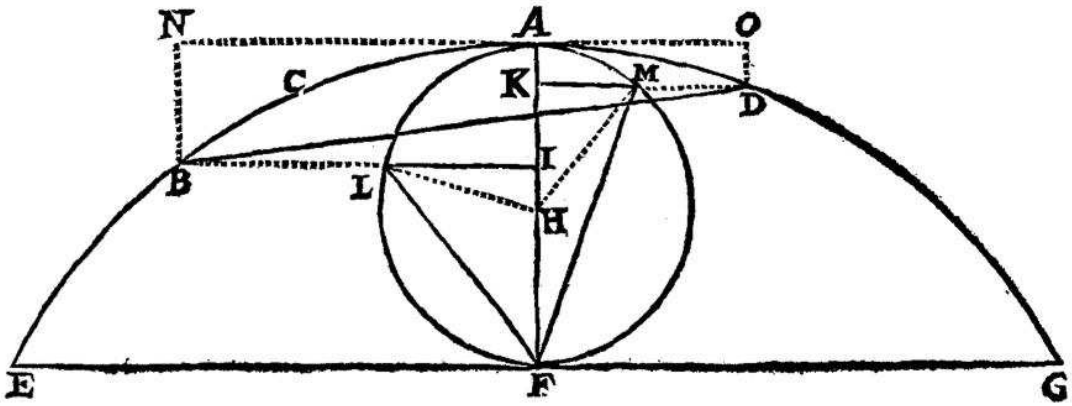
Pag. 66. *Quarrer l'éspace cycloidal. &c.* Quel abus des Séries ! On trouve bien l'aire sans elles, & même très-simplement sans calcul.
(Joh. Bernoulli, *Remarques sur le livre intitulé "Analyse des infinimens petits..." par Mr. Stone, de la Soc. Roy. de Lond., 1735*, *Opera* IV, p. 175)

Mem. in the AE from 1699 (*Opera* I, p. 322-327): *Cycloidis primariæ Segmenta innumera Quadraturam recipientia ; aliorumque ejusdem spatiorum quadrabilium determinatio : post varias illius fortunas nunc primum detecta.*

(French version *"Quadrature d'une infinité de segments de la cycloïde ordinaire ; détermination nouvelle d'autres espaces quarrables après des fortunes diverses"* is presented to the l'Académie des sciences in Paris on July 11, 1699, publ. in l'*Histoire de l'Académie royale des sciences* 1699, p. 134-139)

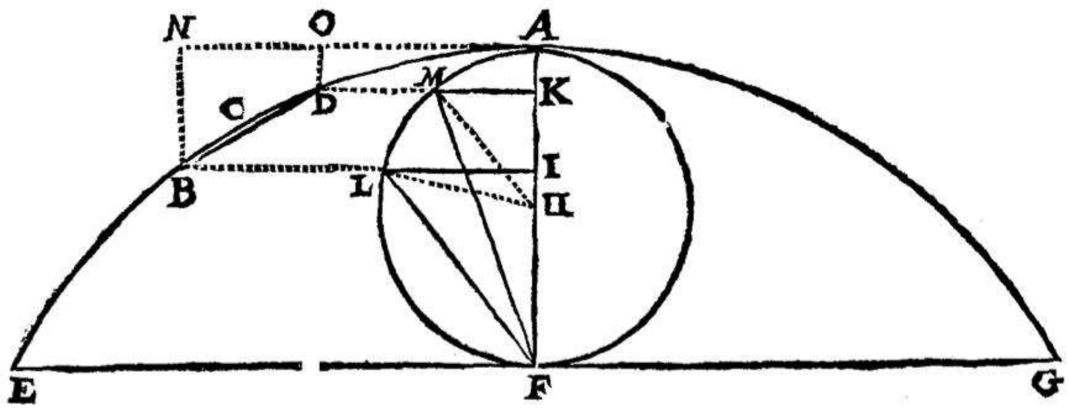
Johann starts by writing "Præcise centum sunt anni (anno nimirum 1599 ...)", and presents, after a long historic overview, **three new Theorems:**

Theorem 1.



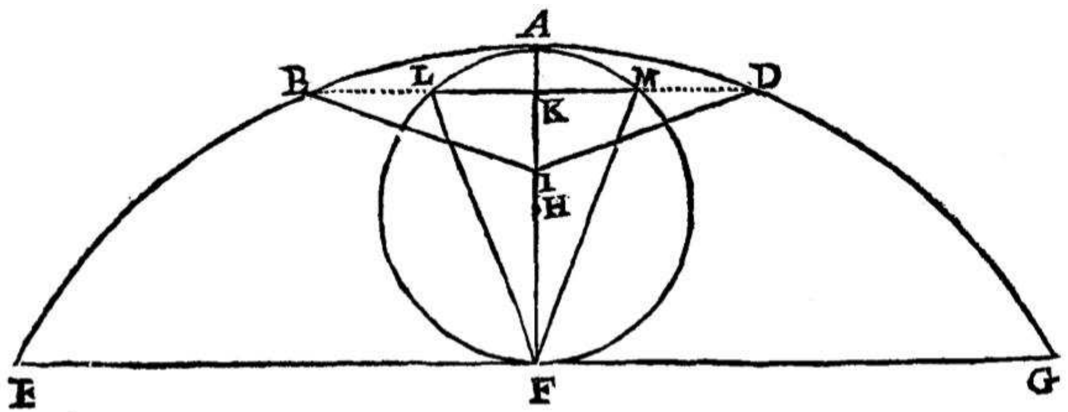
If $AK = IH$ then area $BCDB =$ tr. $LFI + MFK$

Theorem 2.



If $AK = IH$ then area $BCDB =$ tr. $LFI - MFK$

Theorem 3.



If $AK = IH$ then area $IBADI =$ isosc.tr. LFM

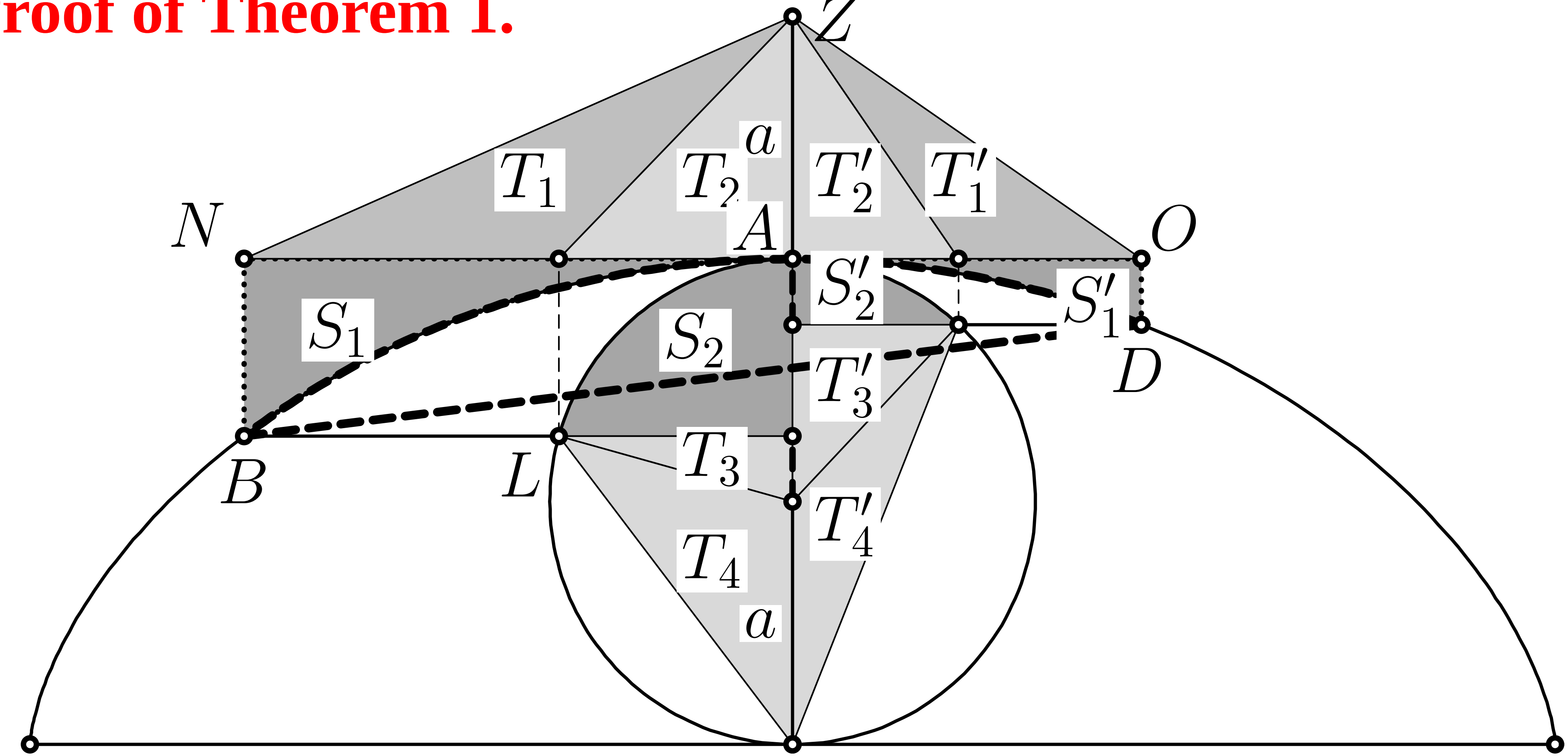
Johann's Proof:

Ut ad rem veniam: Sit Cycloïs primaria E A G, cujus baſis EG, axis AF, circulus generator ALF: Ductæ pro lubitu duæ applicatæ IB & KD, illa a centro H, hæc a vertice A æqualiter diſtantes, determinabunt in Cycloide duo puncta B & D, quibus junctis recta BD, ductiſque LF & MF: Dico *Segmentum Cycloidicum BCDB fore quadrabile, æquale nimirum (in Fig. I.) ſummæ triangulorum rectilineorum LFI + MFK, & (in Fig. II.) differentiæ eorumdem LFI — MFK*: quod ſic demonſtro. Concipiantur ductæ HL, HM; item NAO, parallela baſi EG; & BN, DO parallelæ axi AF. Applicatis IB, KD (*Fig. I.*) exiſtentibus ad partes oppoſitas, erit ſegm. BCDB = trapezio BNOD demtis duobus trilineis ANB & AOD; Eſt autem trapez. BNOD = ($\frac{1}{2}$ BN + $\frac{1}{2}$ DO) in NO = [ob HI = AK] $\frac{1}{2}$ HA in NO = $\frac{1}{2}$ HA in NA + $\frac{1}{2}$ HA in OA. Jam ex natura Cycloidis, $\frac{1}{2}$ HA in NA = $\frac{1}{2}$ HA in (arcum AL + rect. LI) = ſectori LHA + triang. LHF = ſectori LFA; eodem modo demonſtratur $\frac{1}{2}$ HA in OA = ſect. MFA. Hinc trapez. BNOD = ſect. LFA + ſect. MFA. Quoniam vero iterum, ex natura Cycloidis, trilin. ANB = ſegm. circ. AIL, & trilin. AOD = ſegm. circ. AKM; ablatis AIL & AKM, a ſectoribus AFL & AFM, remanebunt duo triangula rectilinea LFI + MFK = trapez. BNOD — ANB — AOD = ſegmento cycloidis BCDB. *Q. E. D.* Porro applicatis IB, KD (*Fig. II.*) exiſtentibus ad partes eaſdem; ſegm. BCDB = trapez. BNOD — trilin. ANB + trilin. AOD. Jam prioribus veſtigiis inſiſtendo habetur trapez. BNOD = $\frac{1}{2}$ HA in NA — $\frac{1}{2}$ HA in OA = ſect. LFA — ſect. MFA; ſurrogatis igitur, loco trilineorum ANB, AOD, ipſis æqualibus ſegm. circ. AIL, AKM, prodibit ſector LFA — ſect. MFA — AIL + AKM, id eſt triang. rectil. LFI — MFK = ſegmento Cycloid. BCDB. *Q. E. D.*

COROLL. I. Si puncta K & I coincidunt, manifeſtum eſt haberi [Fig. I.] ſegmentum rectum *Hugenianum* = trian-

TAB. V N°. LVII *Fig. 1. Fig. 2.*

Proof of Theorem 1.



$AK = IH \Rightarrow \frac{NB+OD}{2} = \frac{a}{2} \Rightarrow$ trap $BNOD = T_1 + T_2 + T_2' + T_1'$

Insert $\underbrace{T_1 = S_2 + T_3\,,\quad T_1' = S_2' + T_3'}_{\text{HM1 and Archimedes}}$ $\underbrace{T_2 = T_4\,,\quad T_2' = T_4'}_{\text{Eucl. I.41}}$

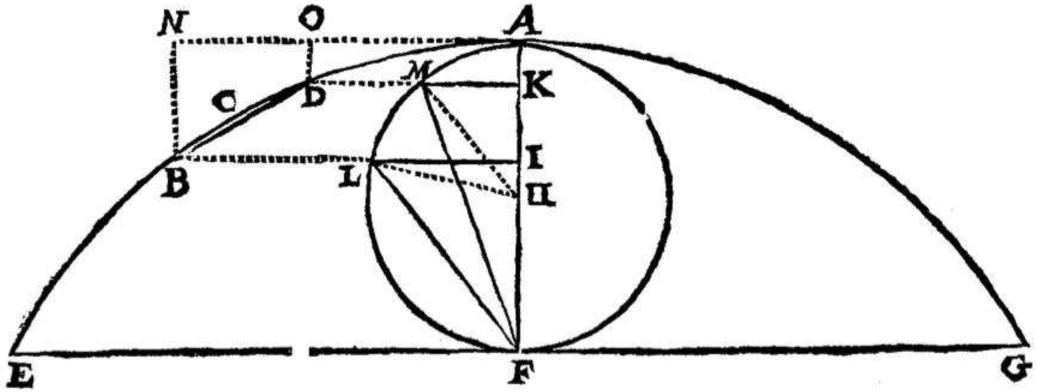
hence Trapezium = Carrot. Subtract from both $S_1 = S_2$ and $S_1' = S_2'$ (**Lemma 2**) gives $\boxed{\text{Segm.}_{BADB} = T_3 + T_4 + T_3' + T_4'}$.

Proof of Theorem 2.

$NO = NA - AO \quad \Rightarrow$

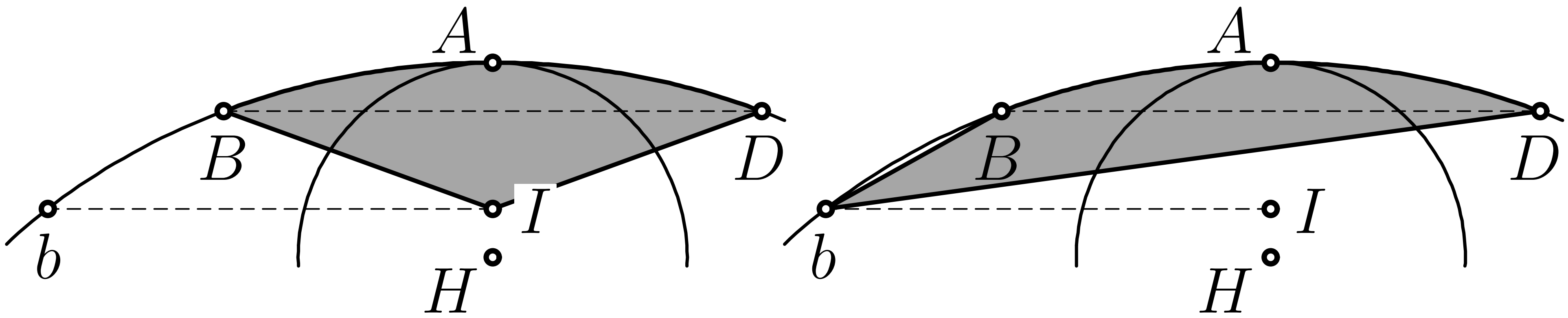
$\text{Trap}_{BDON} = T_1 + T_2 - T_2' - T_1'$

hence



$$\text{Segm.}_{BDB} = \text{Trap}_{BDON} - S_1 + S_1' = T_4 + T_3 - T_4' - T_3' \, .$$

Proof of Theorem 3.



Move the point I horizontally to b on the cycloid.

$$\text{area tri.} IBD = bBD \quad \Rightarrow \text{area sect.} IBAD = bBAD = \ldots$$

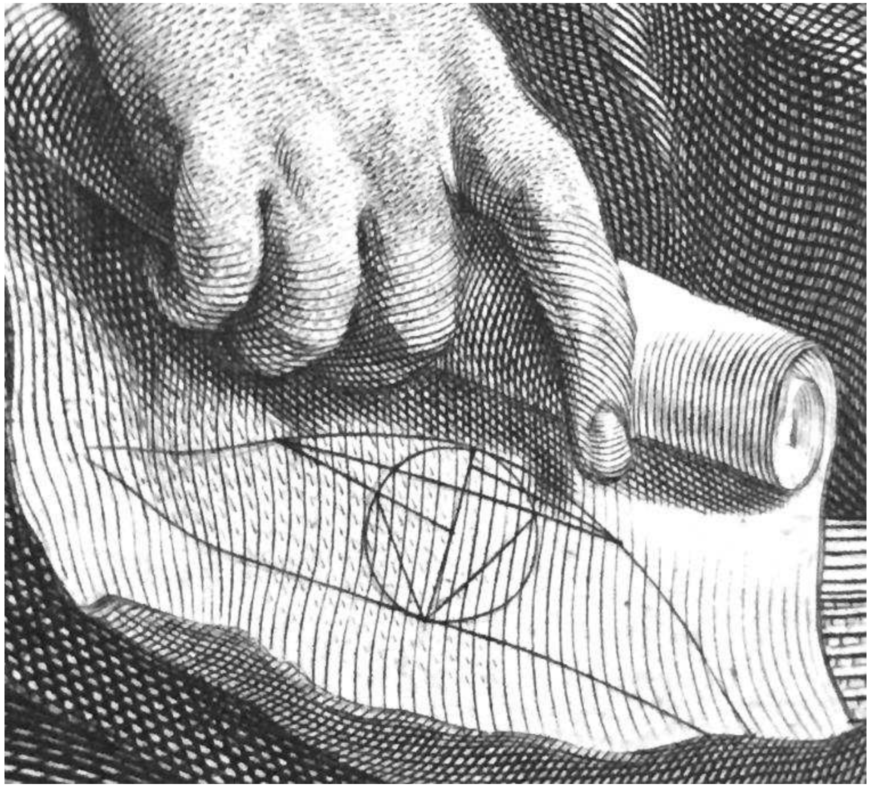
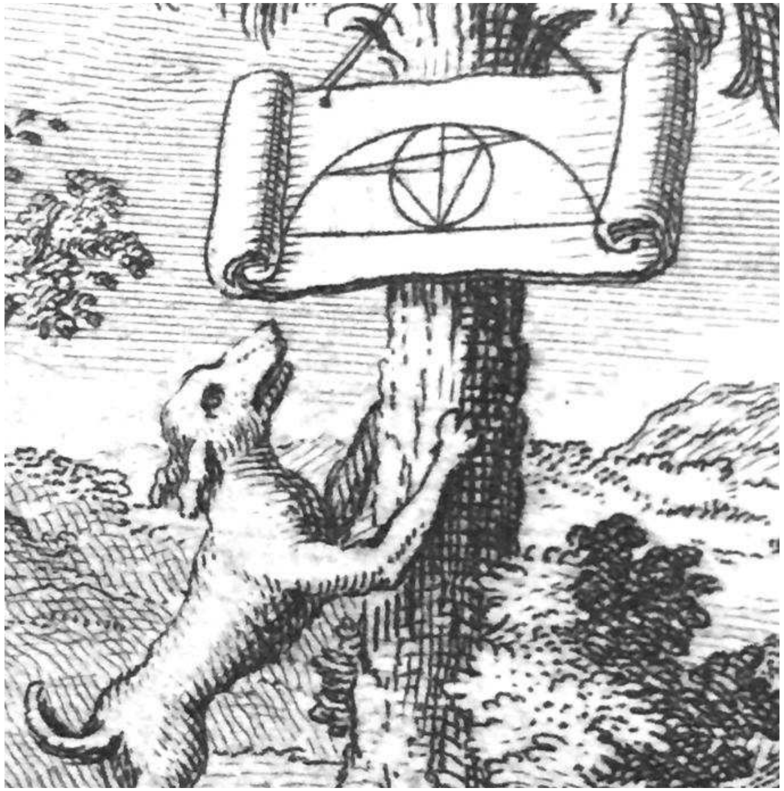
$$\ldots = \text{Res.Thm.1} - \text{Res.Thm.2} = 2(T_4' + T_3').$$

Theorem 1 for $K = I$ is the result of Huygens and for $I = A$ (and therefore $K = H$) is that of Leibniz.

At the end of his article sent to Paris, Johann Bernoulli stated that, whenever the “démonstration synthetique” of his general results “aura eu le bonheur de plaire à l’Académie”, he would transmit also his analytic calculations, which were at the origin of their discovery.

Conclusion

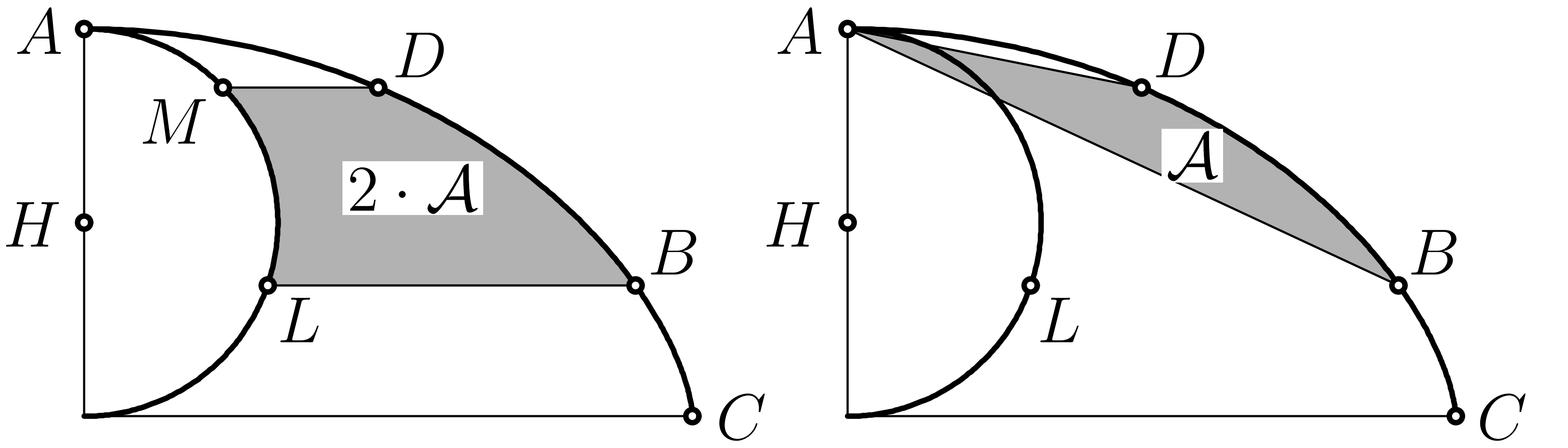
Which of the many theorems on the cycloid was then **the** Theorem for which Johann was particularly proud and on which he based his posterity? Let us have a closer look to the pictures:

We recognize clearly the picture for **Theorem 1** of the *Acta Eruditorum* from July 1699. Precisely 100 years after the discovery of the cycloid by Galilei, after a century of struggles by the most eminent mathematicians (Descartes, Torricelli, Roberval, Pascal, Huygens and Leibniz), he was able to generalize two particular results of the last two of them and realized an unexpected discovery in a long standing tradition.

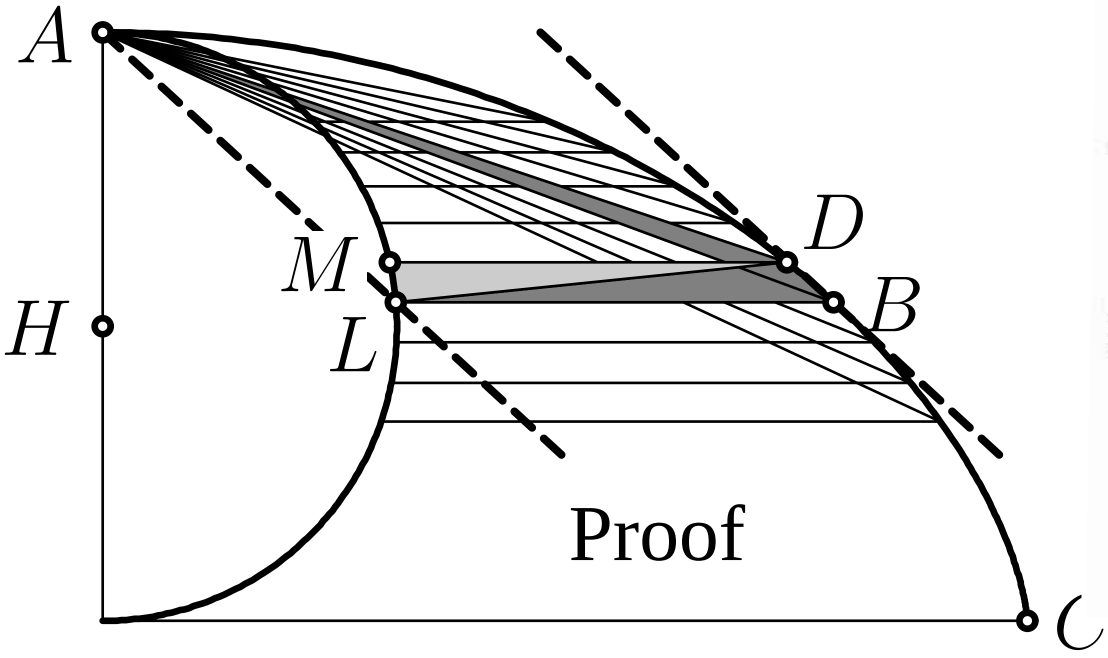
Jakob Bernoulli's reaction. This discovery of his brother was apparently a bitter pill for Jakob; three months later (September 1699) he published an article in the *Acta Eruditorum*, sept. 1699, p. 427-428, also Johann's *Opera* [I, p. 328-329]: *Quadratura zonarum cycloïdalium demonstrata (Quadrature démontrée des zones cycloïdales)*, not mentioning with a word the result of his brother, and by proving

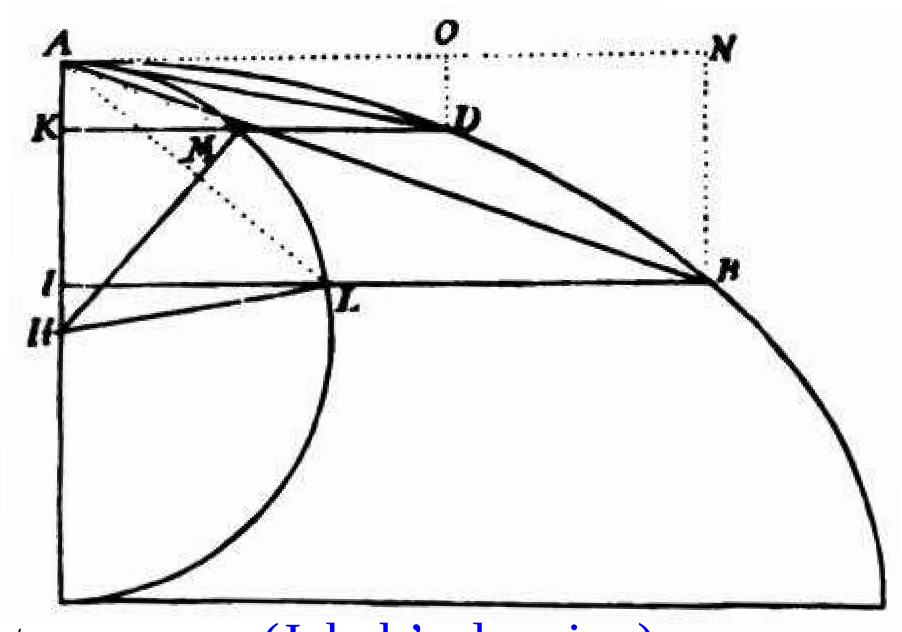
Theorem. *Let B and D be points on the same half of a cycloid with summit A, then the area of the space $BDML$ is twice the area of the cycloidal triangle DBA:*



Jakob's **Proof** consisted in a page of analytical calculations and was later modified by Gabriel Cramer (1704-1752) .

But we can boil it down to Eucl. I.41 and Eucl. I.35:





(Jakob's drawing)

Jakob terminates his article by writing: "Methodum vero tam facilem haud alia fini pandere volui, quam ut Frater, exemplo meo, ad paria præstanda incitatus, mei quoque Problematis Isoperimetrici promissam analysis tandem aliquando nobis impertiat [I developed this really easy result for no other reason that my brother follows my example and reveals after all his promises his solution of my problem about isoperimetric curves]"

This last sentence should remind the readers of the *Acta* of the only — and painful — defeat of Johann in their rude rivalty.

3. Trigonometric Functions.

Euler E246, written 1754: “Logarithmi quidem statim primis ... iisque imprimis calculus exponentialium cuius inuentionem Cel. *Joh. Bernoulli b. m.* iure sibi vindicanerat...”

$$x = t + \sin t\,, \quad y = 1 + \cos t$$

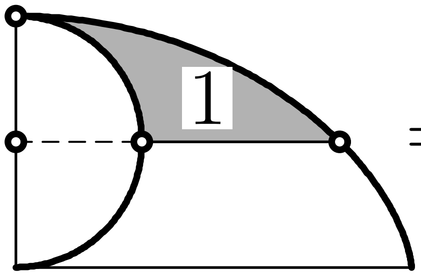
$$\frac{dy}{dx} = \frac{-\sin t}{1+\cos t} = \tan\frac{t}{2}$$

$$ds = \sqrt{(1+\cos t)^2 + \sin^2 t} = 2\cos\frac{t}{2}$$

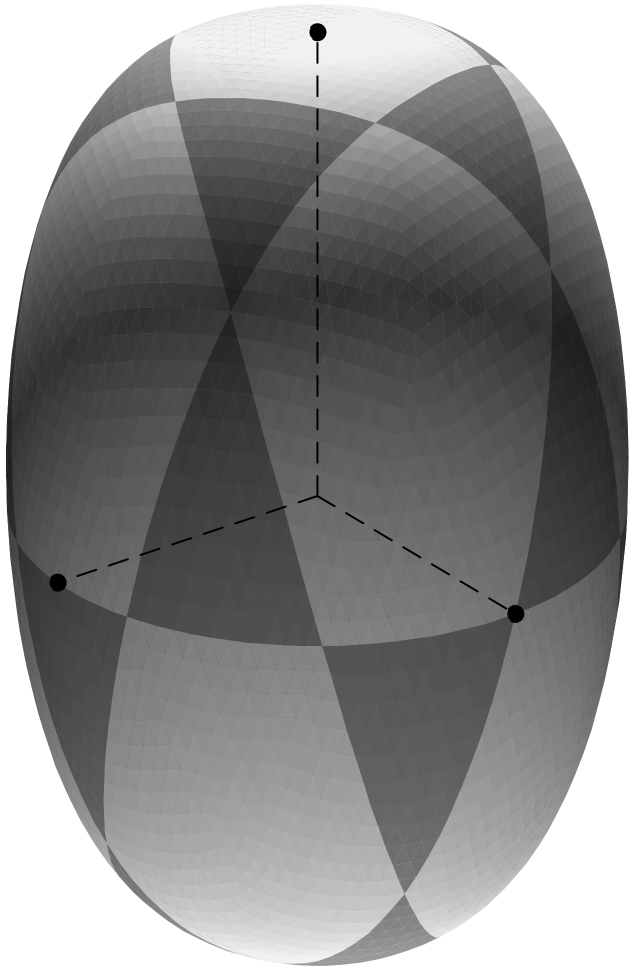
$$s = 2\int_{-\pi}^{\pi} \cos\frac{t}{2}\,dt = 8$$

$$\mathcal{V} = \pi\int_{-\pi}^{\pi} y^2\,dx = \int_{-\pi}^{\pi}(1+\cos t)^3\,dt$$

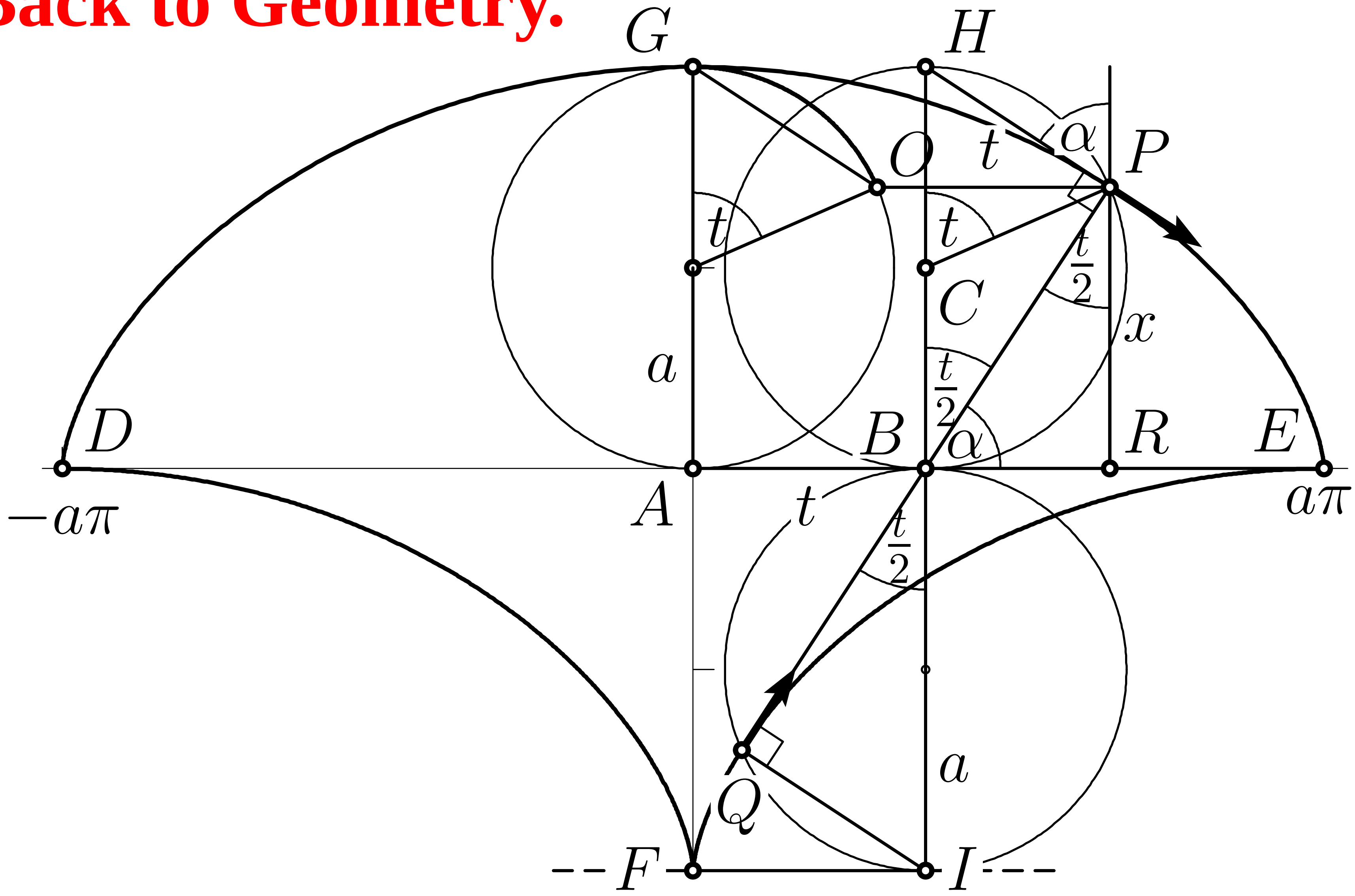
$$= (1..3..3..1) = 2\pi^2 + 0 + 3\pi^2 + 0 = 5\pi^2$$



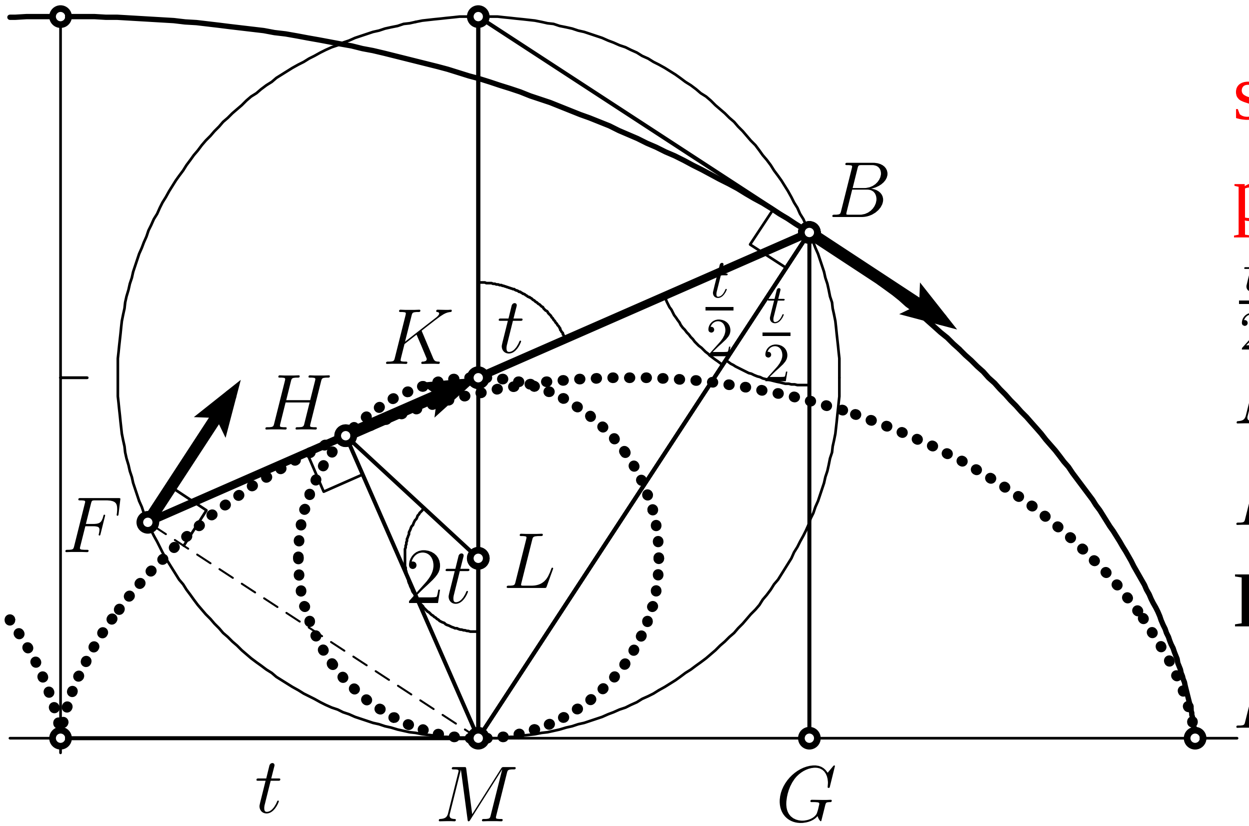
$$= \int_0^{\frac{\pi}{2}} t\cdot\sin t\,dt$$ (Leibniz 1674)



4. Back to Geometry.



At every moment the circles HB and BI rotate around the base point B resp. I. Just look ate the velocities of P and Q **everything clear!!** see also $x = 2a\sin^2\alpha$ (Brachystochrone)



same idea for
position of caustic !!

$\frac{t}{2} = \frac{t}{2}$; $FB =$ refl. ray

$MHB = \llcorner$; H no lat.vel.

H is position of caustics

Eucl. III.20; $MLH = 2t$

$HHH\ldots$on cycloid.

It is not easy to use the *geometric* method to discover things, it is very difficult, but the elegance of the demonstrations after the discoveries are made, is really very great. The power of the *analytic* method is that it is much easier to discover and to prove things, but not in any degree of elegance. There is a lot of dirty paper with x-es and y-s and crossed out cancellations and so on ... (laughers).

(R. Feynman, lecture of march 13, 1964, 35th minute.)

Grazie tanto