Stochastic processes and their applications in Economics

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32. Schweizerischer Tag über Mathematik und Unterricht
Scuola Cantonale di Commercio (SCC), Bellinzona
September 14th, 2022

Introduction

- A stochastic process is a collection of random variables Y_t indexed by time t
- Beautiful mathematical theory with relevant applications in many domains: physics, chemistry, biology (e.g. Brownian motion), meteorology, engineering sciences, ... and economics
- Why are economists (and social scientists) interested in stochastic processes?

Introduction

- Economics deals with **intertemporal decisions** under **uncertainty**, i.e. with time and risks!
- An agent's decision today has an effect today and in the future
- In real life situations, the agent's opportunity set is subject to uncertainty (randomness)
- Example: decision to buy a new house
- Stochastic processes yield the mathematical language to describe risks evolving in time

Outline

Stochastic processes: definition

- An underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$

- A stochastic process is a collection of random variables (or vectors, or matrices)

$$\{Y_t \ : \ t \in \mathcal{T}\}$$

valued in some space $\mathcal{Y}$

- The index set may be continuous e.g. $\mathcal{T} = [0, \infty)$ or discrete e.g. $\mathcal{T} = \mathbb{N}$

- For any given $t \in \mathcal{T}$, we have a random variable (vector, matrix) $Y_t \ : \ \Omega \rightarrow \mathcal{Y}$

- For any given $\omega \in \Omega$, we have a time series of realizations

White Noise processes

- A White Noise is a stochastic process with Y_t independent and identically distributed (i.i.d.) across $t = 0, 1, 2, ...$
- Gaussian White Noise: Y_t i.i.d. $N(0, \sigma^2)$, i.e. $\mathcal{Y} = \mathbb{R}$ and

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} y^2\right)$$

- Bernoulli process: Y_t i.i.d. $B(1, p)$, i.e. $\mathcal{Y} = \{0, 1\}$ and $\mathbb{P}(Y_t = 1) = p$

Markov process

- Serial dependence means that the density of Y_t depends on the past realizations $Y_{t-1}, Y_{t-2}, ...$

$$f(Y_t | Y_{t-1}, Y_{t-2}, ...)$$

- For a Markov process of order p, only the p most recent lags matter

$$f(Y_t | Y_{t-1}, Y_{t-2}, ...) = f(Y_t | Y_{t-1}, Y_{t-2}, ..., Y_{t-p})$$

Autoregressive process

Let $\mathcal{Y} = \mathbb{R}$. In an **Auto-Regressive process of order** p there is a linear additive effect from p lags

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t$$

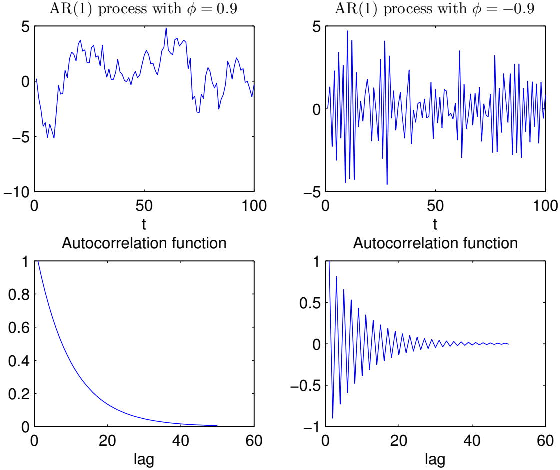
where $\varepsilon_t \sim WN(0, \sigma^2)$

Example: AR(1) process

$$\begin{aligned} Y_t &= \phi Y_{t-1} + \varepsilon_t \\ &= \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + ... \end{aligned}$$

if $|\phi| < 1$.

AR(1) process



AR(1) process with $\phi = 0.9$
AR(1) process with $\phi = -0.9$
5
0
−5
−10
0
50
100
t
Autocorrelation function
Autocorrelation function
1
0.8
0.6
0.4
0.2
0.5
−0.5
−1
20
40
60
lag

Markov chains

Definition 1

The stochastic process Y_t is a (time homogenous) Markov chain if:

(i) it has a discrete state space $\mathcal{Y} = \{1, 2, ...\}$, and

(ii) satisfies the Markov property (of order 1):

$$\mathbb{P}(Y_t = j | Y_{t-1}, Y_{t-2}, ...) = \mathbb{P}(Y_t = j | Y_{t-1})$$

Here we focus on Markov chains with finite state space $\mathcal{Y} = \{1, 2, ..., J\}$

Transition matrix

The distribution of an homogenous Markov chain (Y_t) is characterized by:

(1) The transition matrix $P = [p_{i,j}]$ where

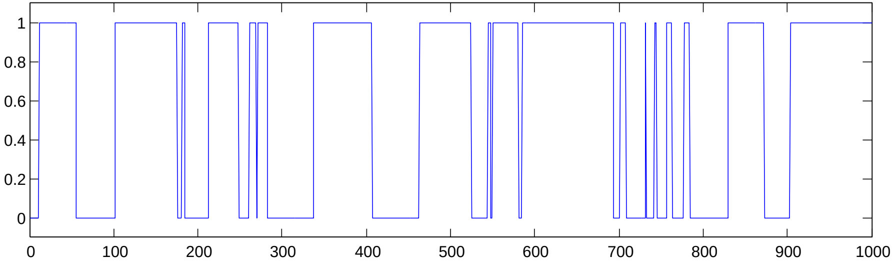
$$p_{i,j} = \mathbb{P}[Y_t = i | Y_{t-1} = j]$$

(2) The initial distribution vector $\mu = [\mu_i]$ where

$$\mu_i = \mathbb{P}[Y_0 = i]$$

Example: A two-state Markov chain

$$P = \begin{pmatrix} 0.95 & 0.025 \\ 0.05 & 0.975 \end{pmatrix}$$



How to simulate a path? $Y_t = \mathbf{1}(U_t \leq p_{1,k})$ with $U_t \sim Unif(0,1)$ and $k = Y_{t-1}$.

Multi-period transition

Let $P^{(h)} = [p_{i,j}^{(h)}]$ denote the transition matrix at horizon $h \geq 1$:

$$p_{i,j}^{(h)} = \mathbb{P}[Y_{t+h} = i | Y_t = j]$$

Theorem 2

We have $P^{(h)} = P^h$, $h \geq 1$.

This is the Chapman-Kolmogorov theorem for Markov chains.

Invariant distribution and stationarity

Definition 3

A vector $\nu = [\nu_1, ..., \nu_J]'$ such that $\nu_j \geq 0$, for all j, and $\sum_{j=1}^{J} \nu_j = 1$ is an invariant distribution of the chain (Y_t) if:

$$P\nu = \nu$$

that is, ν is an eigenvector of matrix P associated with the eigenvalue 1.

If $Y_0 \sim \nu$ then $Y_t \sim \nu$ for all $t \geq 0$.

Irreducible Markov chains

Definition 4

(i) State i is accessible from state j, denoted $j \to i$, if $p_{ij}^{(h)} > 0$ for some $h \geq 0$.

(ii) States i and j communicate, denoted $i \leftrightarrow j$, if both $j \to i$ and $i \to j$.

(iii) A time homogeneous Markov chain is irreducible if any two states communicate: $i \leftrightarrow j$, for all $i, j \in \{1, ..., J\}$.

Example of (non) irreducible chains

Consider the two stochastic matrices:

$$P_1 = \begin{pmatrix} 1/2 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

The Markov chain associated with P_1 is irreducible, while the one associated with P_2 is not!

Ergodic theorem

Theorem 5

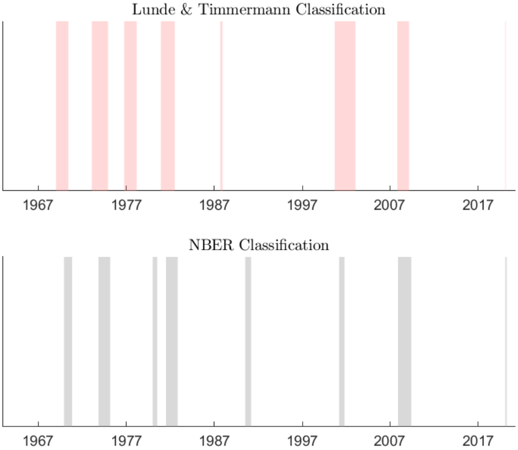
Let (Y_t) be an irreducible homogeneous Markov chain with finite state space and invariant distribution ν. Further, let f be a function on $\{1, 2, ..., J\}$ and $\mathbb{E}[f(Y_t)] := \sum_{j=1}^{J} \nu_j f(j)$. Then, for any initial distribution:

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} f(Y_t) \to \mathbb{E}[f(Y_t)]$$

with probability 1.

Business and stock market cycles

Economic recessions (grey) vs expansions (NBER)
Down-turning (red) vs up-turning stock market (Lunde, Timmermann 2004)



Modeling the business cycles as a Markov chain

Let $S_t =$ either 0 or 1 for the economy state: recession vs boom

Transition matrix

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$$

Estimation (yearly data):

$$\hat{P} = \begin{pmatrix} 0.7550 & 0.0951 \\ 0.2450 & 0.9049 \end{pmatrix}$$

State space models

Hamilton (1989)

y_t an economic time series (e.g. GDP growth rate, or unemployment rate)

Consider the business cycle $S_t = 0/1$ as a **latent** (i.e. unobserved) state

Link y_t to the latent state S_t plus an autocorrelated noise z_t

$$
\begin{aligned}
y_t &= \alpha_0 + \alpha_1 S_t + z_t, \\
z_t &= \phi_1 z_{t-1} + \phi_2 z_{t-2} + \ldots + \phi_p z_{t-p} + \varepsilon_t
\end{aligned}
$$

State space models

Use the model to filter the state

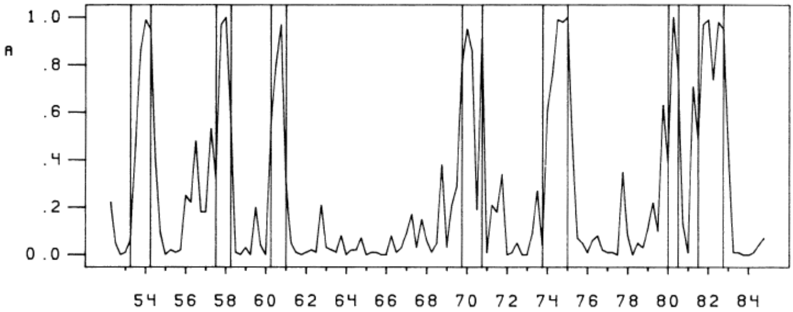
$$\mathbb{P}(S_t = 1 | y_t, y_{t-1}, y_{t-2}, ...)$$

or predict the future state

$$\mathbb{P}(S_{t+1} = 1 | y_t, y_{t-1}, y_{t-2}, ...)$$

Kitagawa-Hamilton filter

Filtered probability of recession state $S_t = 0$ (Hamilton (1989))



Modern approach: common factors from large models

Andreou, Gagliardini, Ghysels, Rubin (ECMA, 2019)
Blue - Quarterly growth rate of Industrial Production (IP) index, USA
Red - Annual growth rate of GDP, USA

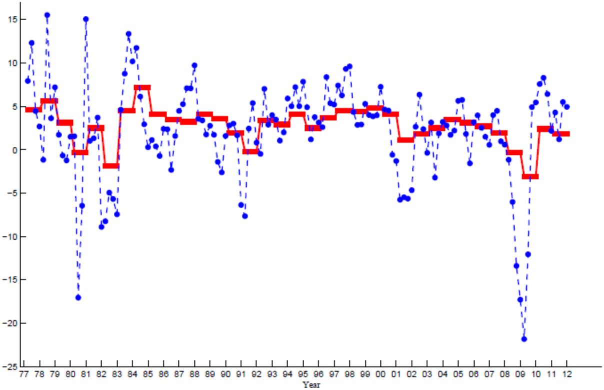
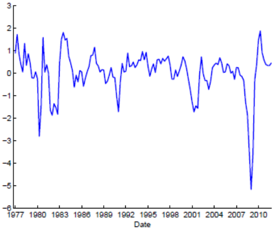
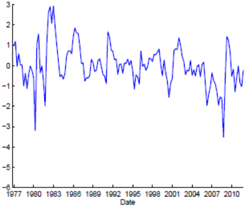


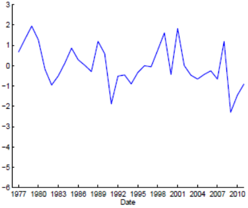
Figure 3: Sample paths of the estimated common and specific factors



(a) Common factor



(b) HF specific factor



(c) LF specific factor

Corporate risks

Let $S_{i,t}$ = either 0 or 1 for risk state of company i in year t (e.g. investment vs speculative rating)

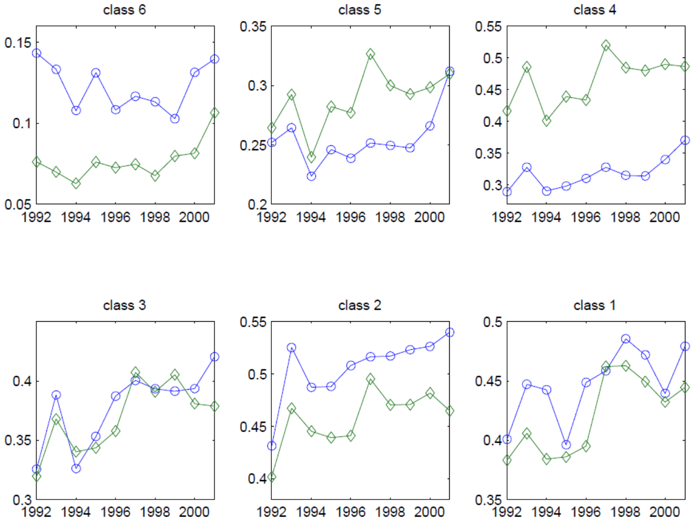
A state vector with n Markov chains

$$S_t = \begin{pmatrix} S_{1,t} \\ S_{2,t} \\ \vdots \\ S_{n,t} \end{pmatrix}$$

How can we model the dependence between the companies?

Rating downgrade probabilities (France: retail, wholesale)

Gagliardini, Gourieroux (JFEC 2005)

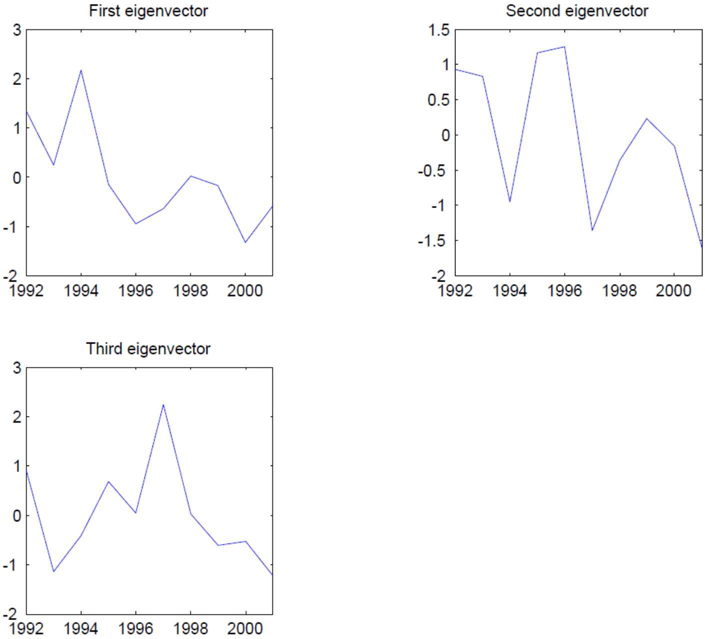


Common factor

A model with a common factor

$$\mathbb{P}(S_{i,t} = 1 | S_{i,t-1} = 1, F_t) = \frac{1}{1 + \exp(a + bF_t)}$$

where F_t follows a AR(1) model $F_t = \phi F_{t-1} + \varepsilon_t$


First eigenvector
3
2
1
0
-1
-2
1992
1994
1996
1998
2000
Second eigenvector
1.5
1
0.5
0
-0.5
-1
-1.5
-2
1992
1994
1996
1998
2000
Third eigenvector
3
2
1
0
-1
-2
1992
1994
1996
1998
2000

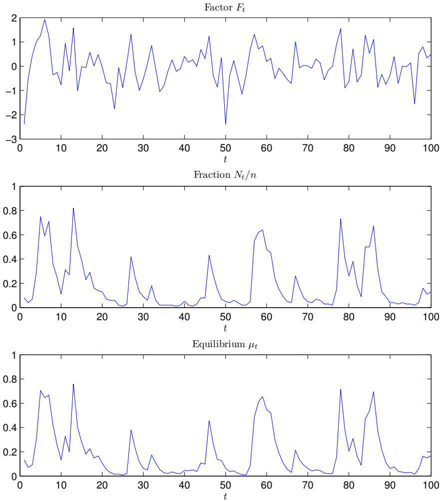
Common factor vs contagion

Add a contagion effect

$$\mathbb{P}(S_{i,t} = 1 | S_{i,t-1} = 1, F_t, N_{t-1}) = \frac{1}{1 + \exp(a + bF_t + cN_{t-1})}$$

where $N_{t-1} = \sum_{i=1}^{n} S_{i,t-1}$ is the number of companies in high risk state at $t - 1$

Gagliardini, Gourieroux (JEDC 2013)


Factor F_t
t
Fraction N_t/n
t
Equilibrium μ_t
t

Concluding remarks

- Stochastic processes can model effectively intertemporal decisions under uncertainty
- Multiple steps in the analysis:
  - Model specification
  - Estimation and testing
  - Prediction
- Do not forget model uncertainty!
- Modern challenges: high-dimensional settings

THANK YOU FOR YOUR ATTENTION!

References

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