Rapsodia triangolare

la combinatoria da Pascal ai giorni nostri

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(IDSIA USI/SUPSI, Lugano)

Corso CMSI "Pascal a 401 anni dalla nascita"
Bellinzona, 1 marzo 2024.

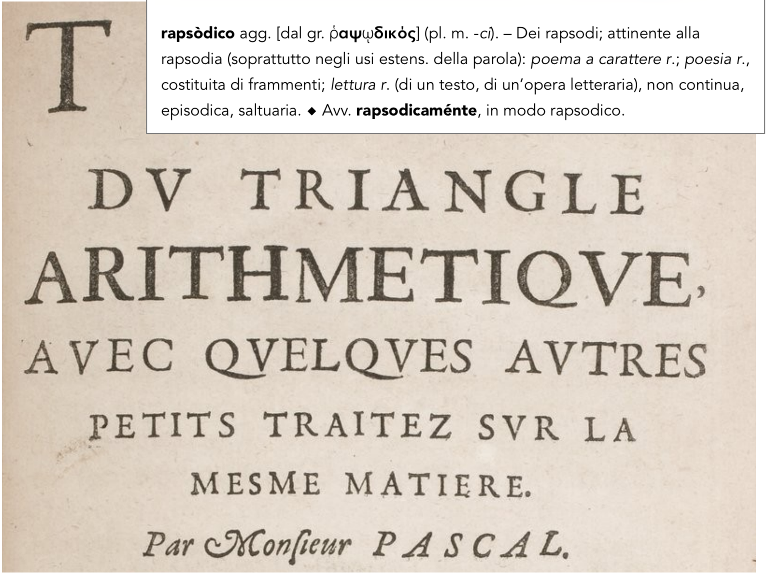
TRAITE
DV TRIANGLE ARITHMETIQVE,

AVEC QVELQVES AVTRES PETITS TRAITEZ SVR LA MESME MATIERE.

Par Monſieur PASCAL.

ISTITUTO FONDAZIONE TRECCANI CORPORATE EVENTI

rapsòdico agg. [dal gr. **ῥαψῳδικός**] (pl. m. -*ci*). – Dei rapsodi; attinente alla rapsodia (soprattutto negli usi estens. della parola): *poema a carattere r.*; *poesia r.*, costituita di frammenti; *lettura r.* (di un testo, di un'opera letteraria), non continua, episodica, saltuaria. ◆ Avv. **rapsodicaménte**, in modo rapsodico.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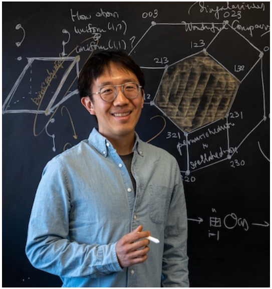
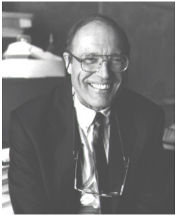
"Fil rouge"

Blaise Pascal
1623 – 1662



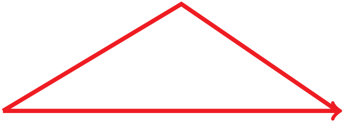
June Huh
*1983

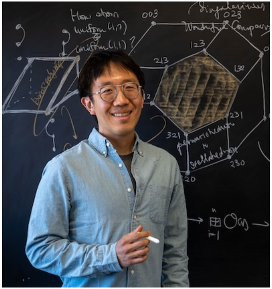
"Triangle rouge"



Gian-Carlo Rota
1932 – 1999

Blaise Pascal
1623 – 1662





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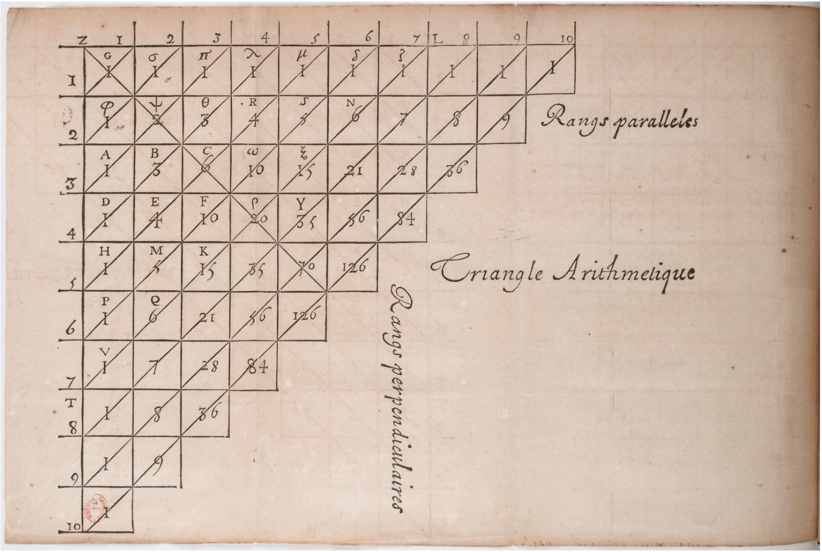
Il triangolo aritmetico



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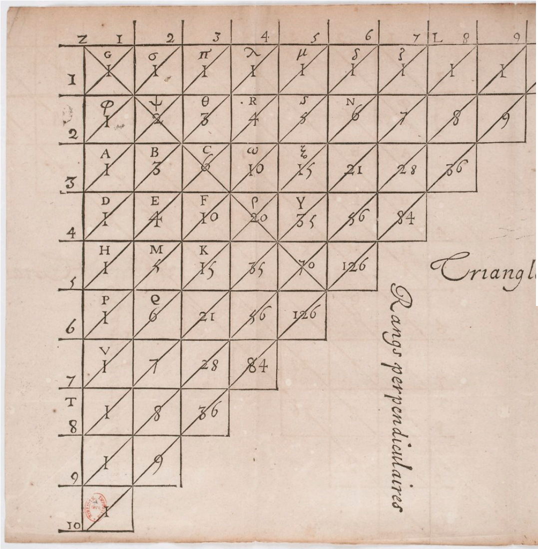

Definizione

$$\binom{n}{0} := \binom{n}{n} := 1$$

$$\binom{n}{k} := \binom{n-1}{k} + \binom{n-1}{k-1}$$

per $n > k > 0$

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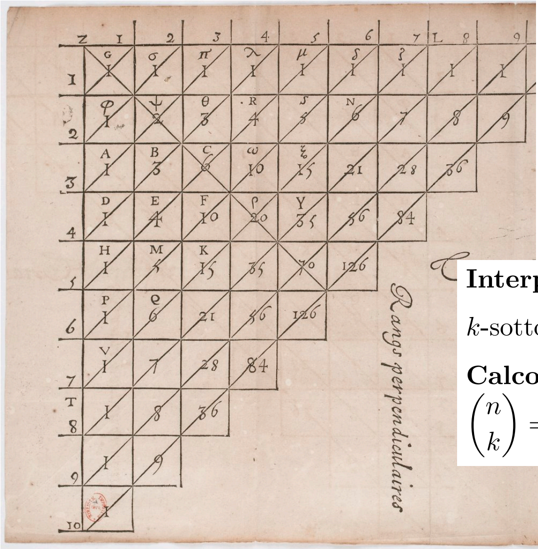

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Interpretazione: numero di k-sottoinsiemi di un n-insieme.

Calcolo:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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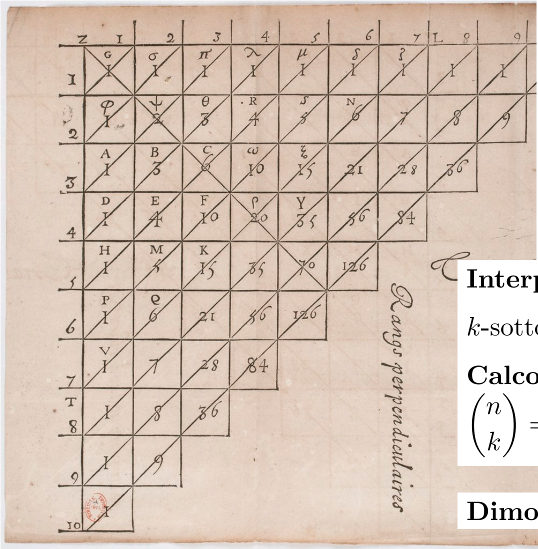

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Dimostrazioni combinatorie!

Arithmetiques, ie passe à diuers vsages de ceux dont le generateur est l'vnité; c'est ce qu'on verra dans les traictez suiuans. Mais i'en laisse bien plus que ie n'en donne; c'est vne chose estrange combien il est fertile en proprietez, chacun peut s'y exercer; I'auertis seulement icy, que dans toute la suite, ie n'entends parler que des Triangles Arithmetiques, dont le generateur est l'vnité.

Un'osservazione ricca di conseguenze

$$\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$$

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$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad \underbrace{6 \quad 15 \quad 20} \quad \underbrace{15 \quad 6 \quad 1}$$

$6 \cdot 20 \leq 15^2$ $\qquad$ $15 \cdot 1 \leq 6^2$

$120 \leq 225$ $\qquad$ $15 \leq 36$

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$$\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$$

Una successione reale $a_0, a_1, a_2, \ldots$ è detta **log-concava** se

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Teorema (Newton). Dato un polinomio a coefficienti reali

$$p(t) = a_0 + a_1 t + \ldots + a_{n-1}t^{n-1} + a_n t^n,$$

se tutti gli zeri di $p(t)$ sono reali, allora la successione a_k è log-concava.

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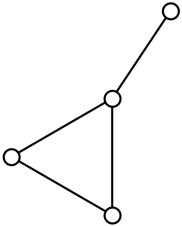
Esempio. $(t+1)^n = \binom{n}{0} + \binom{n}{1}t + \binom{n}{2}t^2 + \ldots \binom{n}{n-1}t^{n-1} + \binom{n}{n}t^n.$

Grafi



V: insieme di *vertici* del grafo G

$v, w \in V$ *adiacenti*

se collegati da un arco.

k**-colorazione** di G:

"coloro i vertici con **al più** k colori; vertici adiacenti hanno colore diverso."

Precisamente:

funzione $\gamma : V \to \{1, \ldots, k\}$ con $\gamma(v) \neq \gamma(w)$ se v, w adiacenti.

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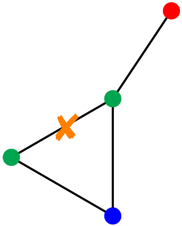

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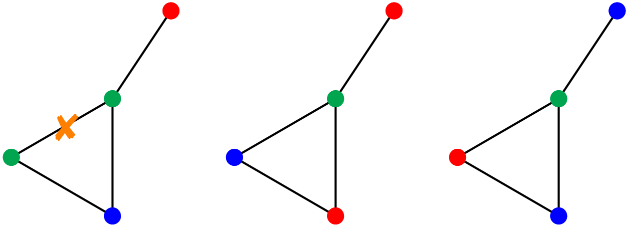

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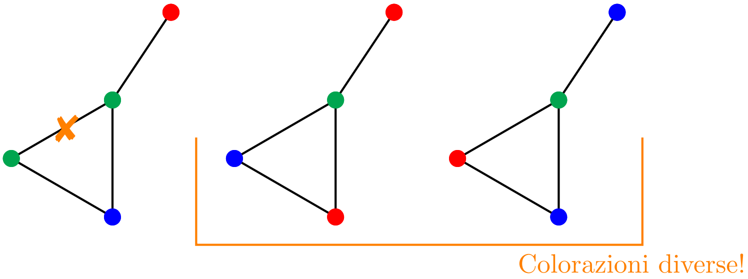

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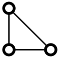
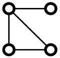
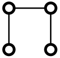
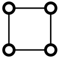
Quante colorazioni?

$$\chi_G(k) = \text{ numero di } k\text{-colorazioni diverse di } G.$$

Esempio/Esercizio.

	$\chi_G(1)$	$\chi_G(2)$	$\chi_G(3)$	$\chi_G(k)$
				
				
				
				

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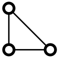
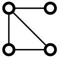
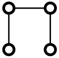
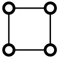
	0	0	$3!$	$k(k-1)(k-2)$
				
				
				

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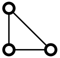
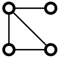
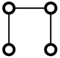
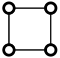
	0	0	$3!$	$k(k-1)(k-2)$
	0	0	$3! \cdot 2$	$k(k-1)^2(k-2)$
				
				

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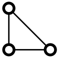
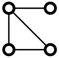
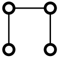
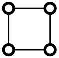
	0	0	$3!$	$k(k-1)(k-2)$
	0	0	$3! \cdot 2$	$k(k-1)^2(k-2)$
	0	2	$3 \cdot 2^3$	$k(k-1)^3$
				

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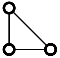
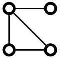
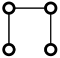
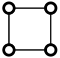
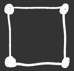
	0	0	$3!$	$k(k-1)(k-2)$
	0	0	$3! \cdot 2$	$k(k-1)^2(k-2)$
	0	2	$3 \cdot 2^3$	$k(k-1)^3$
	0	2	?	?

$\{$ k-colorazioni di  $\}$

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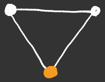
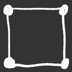
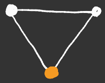
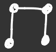
$\{$ k-colorazioni di  $\}$

{ k-colorazioni di  } = { k-colorazioni di  } \ { k-colorazioni di  }

χ  $(k) =$ χ  $(k) -$ χ  (k)

$$= k(k-1)^3 - k(k-1)(k-2)$$

$$= \dots = k(k-1)(k^2-3k+3) = k^4 - 4k^3 + 6k^2 - 3k$$

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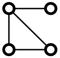
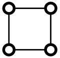
Teorema. $\chi_G(t)$ è sempre un **polinomio** a coefficienti in $\mathbb{Z}$.

(Birkhoff, *Annals* 1912; Whitney, *Annals* 1935)

G	$\chi_G(t)$
	$t^4 - 4t^3 + 5t^2 - 2t = (t-1)^2(t-2)$
	$t^4 - 4t^3 + 6t^2 - 3t = t(t-1)(t^2-3t+3)$

Problema aperto. Quali polinomi si ottengono in questo modo?

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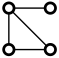
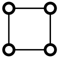
	$t^4 - 4t^3 + 5t^2 - 2t = (t-1)^2(t-2)$
	$t^4 - 4t^3 + 6t^2 - 3t = t(t-1)(t^2 - 3t + 3)$

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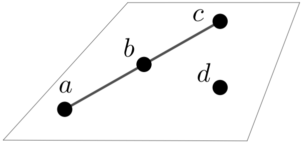
Congettura. Log-concavità dei coefficienti.

Geometria

$\mathcal{P}$: insieme di punti nel piano.

Ad ogni $X \subseteq \mathcal{P}$ assegnamo la

dimensione dello "spazio che genera".



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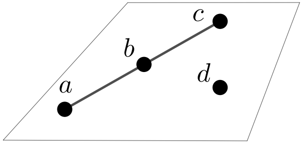


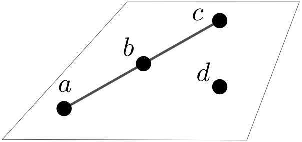
X	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
dim	-1	0	0	0	0	1	1	1	1	1	1	1	2	2	2	2

Geometria

$\mathcal{P}$: insieme di punti nel piano.

Ad ogni $X \subseteq \mathcal{P}$ assegnamo la

dimensione dello "spazio che genera".



X	$\emptyset$	a	b	c	d	ab	ac	ad
dim	-1	0	0	0	0	1	1	1
$t^{1+\mathbf{2}-\dim}$	t^4	t^3	t^3	t^3	t^3	t^2	t^2	t^2

X	bc	bd	cd	abc	abd	acd	bcd	$abcd$
dim	1	1	1	1	2	2	2	$\mathbf{2}$
$t^{1+\mathbf{2}-\dim}$	t^2	t^2	t^2	t^2	t^1	t^1	t^1	t^1

Geometria

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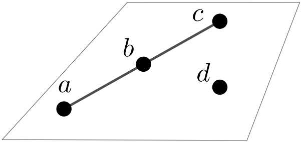

X	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
dim	-1	0	0	0	0	1	1	1	1	1	1	1	2	2	2	$\mathbf{2}$
$t^{1+\mathbf{2}-\mathrm{dim}}$	t^4	t^3	t^3	t^3	t^3	t^2	t^2	t^2	t^2	t^2	t^2	t^2	t^1	t^1	t^1	t^1
$(-1)^{\lvert X\rvert}$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$

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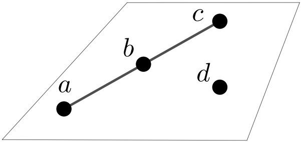

X	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
dim	-1	0	0	0	0	1	1	1	1	1	1	1	2	2	2	**2**
$t^{1+\mathbf{2}-\dim}$	t^4	t^3	t^3	t^3	t^3	t^2	t^2	t^2	t^2	t^2	t^2	t^2	t^1	t^1	t^1	t^1
$(-1)^{\lvert X\rvert}$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$

$$\chi(t) = t^4 - 4t^3 + 5t^2 - 2t^1$$

Geometria

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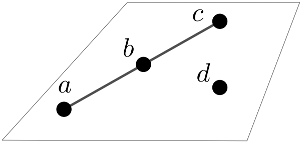
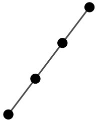


X	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
dim	-1	0	0	0	0	1	1	1	1	1	1	1	2	2	2	**2**
$t^{1+\mathbf{2}-\dim}$	t^4	t^3	t^3	t^3	t^3	t^2	t^2	t^2	t^2	t^2	t^2	t^2	t^1	t^1	t^1	t^1
$(-1)^{\lvert X\rvert}$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$

$$\chi(t) = t^4 - 4t^3 + 5t^2 - 2t^1$$

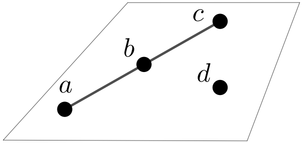
DIY:



Geometria

$\mathcal{P}$: insieme di punti nel piano.

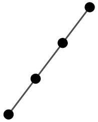
Ad ogni $X \subseteq \mathcal{P}$ assegnamo la *dimensione* dello "spazio che genera".



X	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
dim	-1	0	0	0	0	1	1	1	1	1	1	1	2	2	2	**2**
$t^{1+\mathbf{2}-\dim}$	t^4	t^3	t^3	t^3	t^3	t^2	t^2	t^2	t^2	t^2	t^2	t^2	t^1	t^1	t^1	t^1
$(-1)^{\lvert X \rvert}$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$

$$\chi(t) = t^4 - 4t^3 + 5t^2 - 2t^1$$

DIY:



$$\chi(t) = t^3 - 4t^2 + 3t$$

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$\mathcal{P}$: insieme di punti nel piano.

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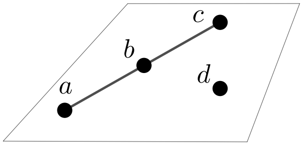
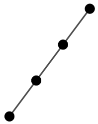


X	$\emptyset$	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
dim	-1	0	0	0	0	1	1	1	1	1	1	1	2	2	2	**2**
$t^{1+\mathbf{2}-\dim}$	t^4	t^3	t^3	t^3	t^3	t^2	t^2	t^2	t^2	t^2	t^2	t^2	t^1	t^1	t^1	t^1
$(-1)^{\lvert X \rvert}$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$

$$\chi(t) = t^4 - 4t^3 + 5t^2 - 2t^1$$

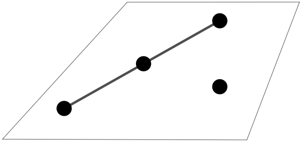
DIY:



$\chi(t) = t(t-1)(t-3)$

Geometria combinatoria

Regole:



Due linee si incontrano in al più un punto.

Ogni coppia di punti distinti determina una linea

(Due piani si intersecano in al più una linea, &c.)

Geometria combinatoria

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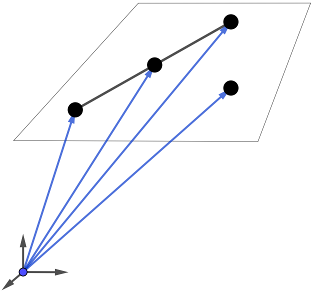
Idea: Vettori su uno "spazio affine".



Geometria combinatoria

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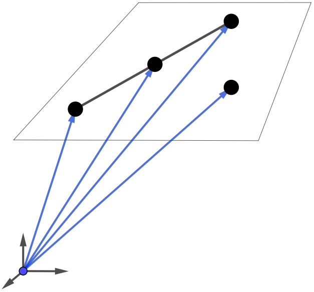
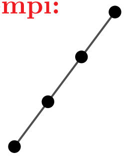


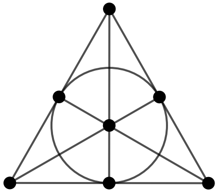
Idea: Vettori su uno "spazio affine".

Esempi:

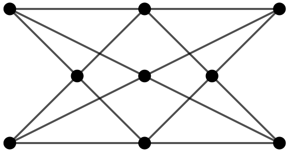


$t^3 - 4t^2 + 3t$

$= t(t-1)(t-3)$

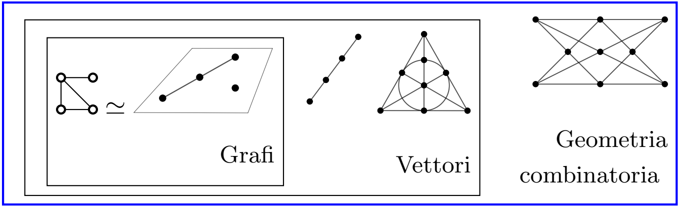


$t^4 - 7t^3 + 14t^2 - 8t$



$t^4 - 9t^3 + 28t^2 - 20t$

Veniamo al dunque

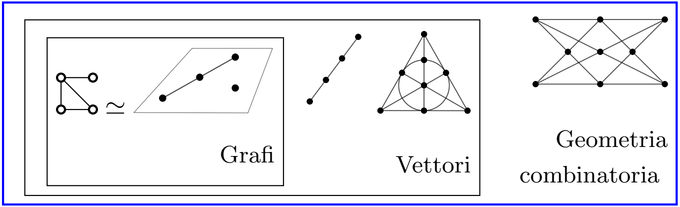


Congettura. (Rota-Heron-Welsh)

In ogni caso "il" polinomio $\chi(t)$ ha coefficienti log-concavi.

Teorema. (Adiprasito-Huh-Katz): Sì.

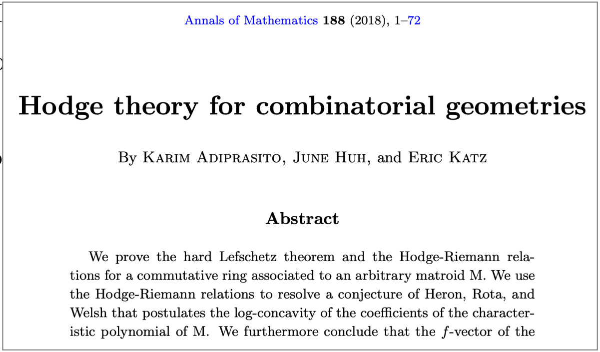
Veniamo al dunque



Congettura. (Rota-H

In ogni caso "il" polino

Teorema. (Adiprasito


Annals of Mathematics **188** (2018), 1–72

Hodge theory for combinatorial geometries

By Karim Adiprasito, June Huh, and Eric Katz

Abstract

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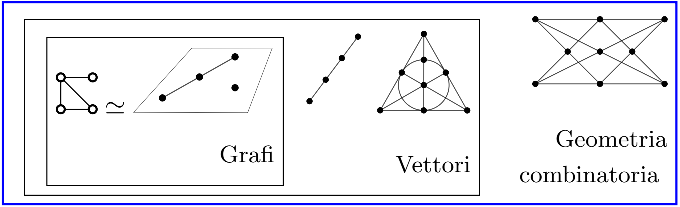
Veniamo al dunque



Matroids

(H. Whitney;
S. MacLane
W. Tutte, &c.)

Congettura. (Rota-H

In ogni caso "il" polino

Teorema. (Adiprasito

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..."SO WHAT?"

- Il triangolo aritmetico è un classico

  "...un libro che non ha mai finito di dire quel che ha da dire." (Calvino)

- I grafi e i loro polinomi (LAM?)

  (1) Palestra per lo studio (e la scrittura) di algoritmi

  (2) Problemi comprensibili senza prerequisiti, "scalabili" a piacere.

  (3) Sperimentazione computazionale; opportunità per lavori originali.

  (`www.sagemath.org`: `chromatic_number()`, `chromatic_polynomial()`...)

- Geometria combinatoria (*matroid theory*)

  Teoria unificante, abbordabile in modo "concreto" senza perdere in generalità.

Delucidazioni e bibliografia:

emanuele.delucchi@supsi.ch

sagemath.org

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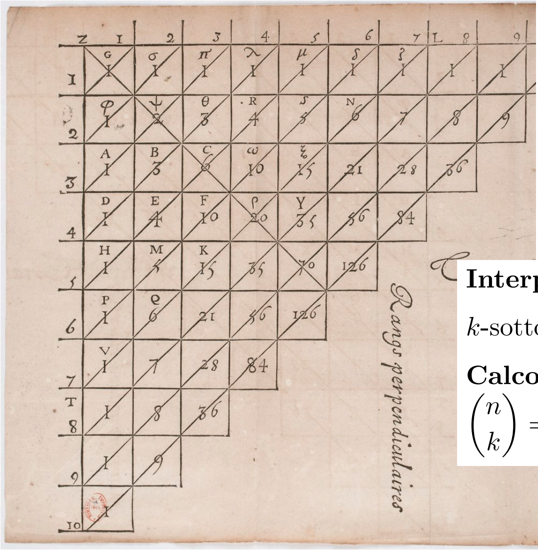

Definizione

$$\binom{n}{0} := \binom{n}{n} := 1$$

$$\binom{n}{k} := \binom{n-1}{k} + \binom{n-1}{k-1}$$

per $n > k > 0$

Interpretazione: numero di k-sottoinsiemi di un n-insieme.

Calcolo:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{5}{3}$$

123

124

125

134

135

145

234

235

245

345

$$\binom{5}{3}$$

$\underline{1}23$

$\underline{1}24$

$\underline{1}25$

$\underline{1}34$

$\underline{1}35$

$\underline{1}45$

$\underline{23}4$

$\underline{23}5$

$\underline{2}45$

$\underline{3}45$

$$3 \cdot \binom{5}{3}$$

$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$
$\underline{1}24$	$1\underline{2}4$	$12\underline{4}$
$\underline{1}25$	$1\underline{2}5$	$12\underline{5}$
$\underline{1}34$	$1\underline{3}4$	$13\underline{4}$
$\underline{1}35$	$1\underline{3}5$	$13\underline{5}$
$\underline{1}45$	$1\underline{4}5$	$14\underline{5}$
$\underline{2}34$	$2\underline{3}4$	$23\underline{4}$
$\underline{2}35$	$2\underline{3}5$	$23\underline{5}$
$\underline{2}45$	$2\underline{4}5$	$24\underline{5}$
$\underline{3}45$	$3\underline{4}5$	$34\underline{5}$

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$\underline{1}23$	$1\underline{2}3$	$12\underline{3}$
$\underline{1}24$	$1\underline{2}4$	$12\underline{4}$
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$\underline{1}34$	$1\underline{3}4$	$13\underline{4}$
$\underline{1}35$	$1\underline{3}5$	$13\underline{5}$
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$\underline{2}34$	$2\underline{3}4$	$23\underline{4}$
$\underline{2}35$	$2\underline{3}5$	$23\underline{5}$
$\underline{2}45$	$2\underline{4}5$	$24\underline{5}$
$\underline{3}45$	$3\underline{4}5$	$34\underline{5}$

$\underline{1}23$	$\underline{2}34$	$1\underline{3}4$	$1\underline{4}5$	$14\underline{5}$
$\underline{1}24$	$1\underline{2}4$	$1\underline{3}5$	$23\underline{4}$	$12\underline{5}$
$\underline{1}25$	$1\underline{2}5$	$2\underline{3}4$	$2\underline{4}5$	$13\underline{5}$
$\underline{1}34$	$1\underline{2}3$	$2\underline{3}5$	$3\underline{4}5$	$23\underline{5}$
$\underline{1}35$	$\underline{2}35$	$\underline{3}45$	$12\underline{4}$	$24\underline{5}$
$\underline{1}45$	$\underline{2}45$	$12\underline{3}$	$13\underline{4}$	$34\underline{5}$

$$3 \cdot \binom{5}{3} = 5 \cdot \binom{4}{2}$$

123	123	123
124	124	124
125	125	125
134	134	134
135	135	135
145	145	145
234	234	234
235	235	235
245	245	245
345	345	345

123	234	134	145	145
124	124	135	234	125
125	125	234	245	135
134	123	235	345	235
135	235	345	124	245
145	245	123	134	345

$$3 \cdot \binom{5}{3} = 5 \cdot \binom{4}{2}$$

123	123	123
124	124	124
125	125	125
134	134	134
135	135	135
145	145	145
234	234	234
235	235	235
245	245	245
345	345	345

123	234	134	145	145
124	124	135	234	125
125	125	234	245	135
134	123	235	345	235
135	235	345	124	245
145	245	123	134	345

$$\binom{5}{3} = \frac{5}{3}\binom{4}{2} = \frac{5 \cdot 4}{3 \cdot 2}\binom{3}{1}$$

$$= \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1}\binom{2}{0} = \frac{5!}{3!\,2!}$$