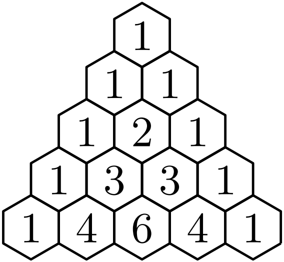
# Du triangle de Pascal à la recherche d'aujourd'hui

Ioan Manolescu

University of Fribourg

1 mars 2024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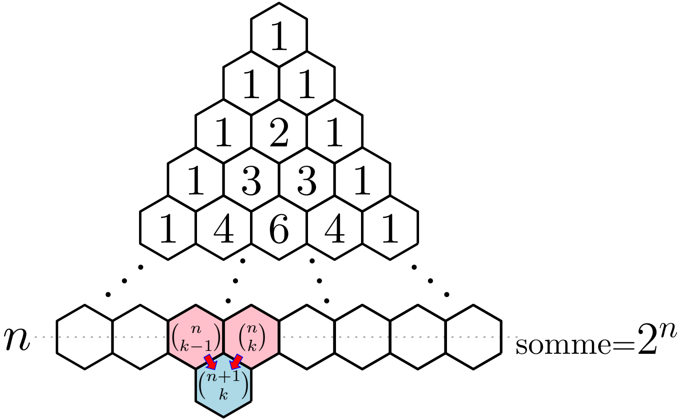
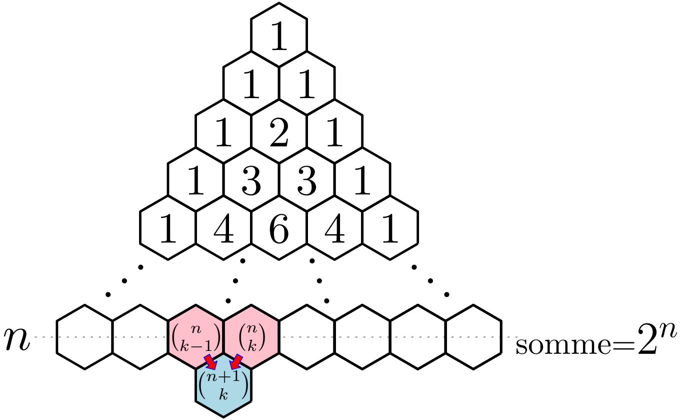
1 1
1 2 1
1 3 3 1
1 4 6 4 1




Construction par recurrence: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

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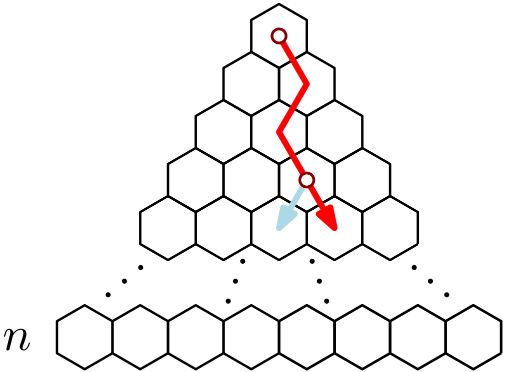
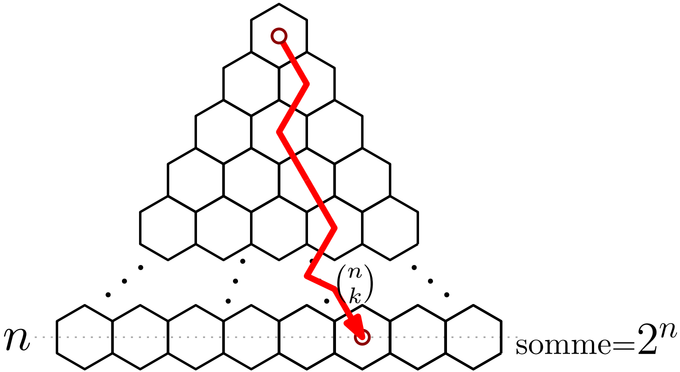
Propriétés: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

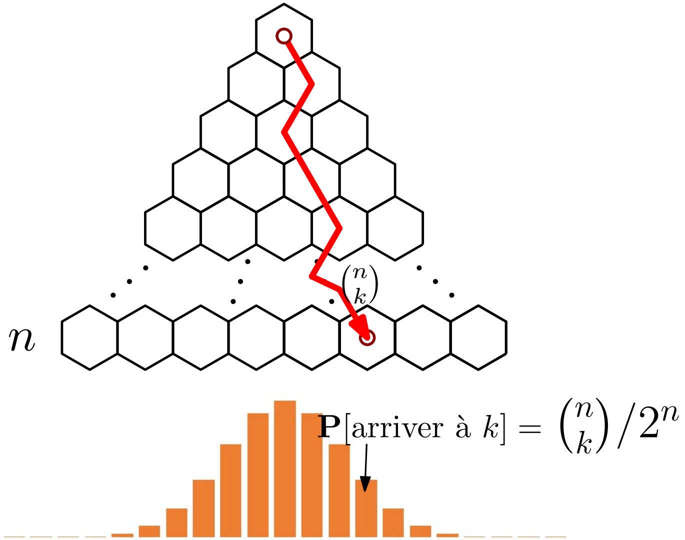
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

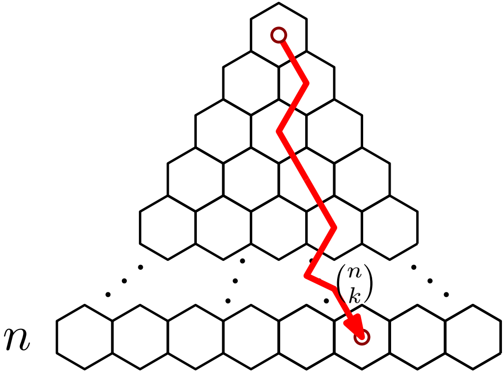

n

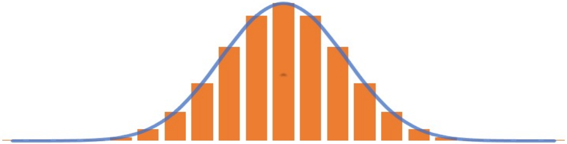



$$\binom{n}{k} = \text{nb. de chemins arrivant à } k\text{, niveau } n$$

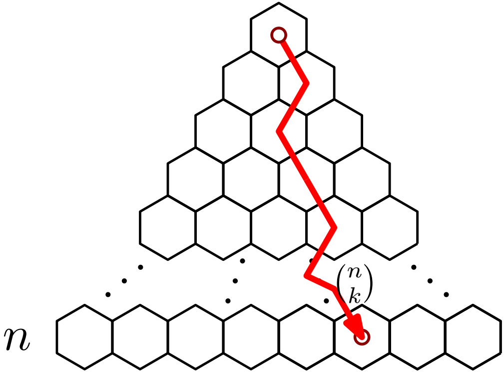
$$= \text{nb. sous-ensemble à } k \text{ éléments d'un ensemble à } n \text{ éléments}$$

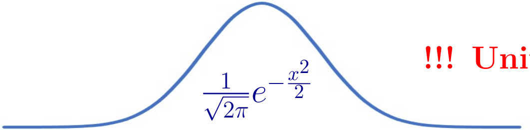

$n$
$\binom{n}{k}$
$\mathbf{P}[\text{arriver à } k] = \binom{n}{k}/2^n$



$$\mathbb{P}[\text{arriver à } \tfrac{1}{2}n + \alpha\sqrt{n}] = \binom{n}{\frac{1}{2}n + \alpha\sqrt{n}}/2^n \sim \frac{1}{\sqrt{n}}\sqrt{\frac{2}{\pi}}\, e^{-2\alpha^2}$$

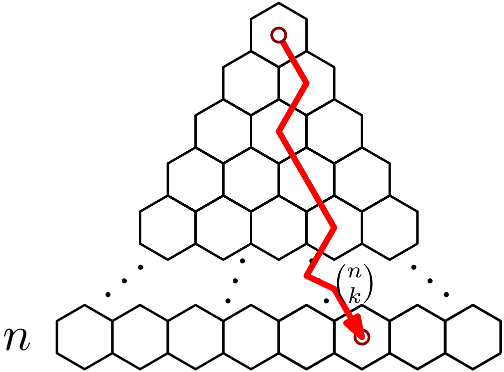


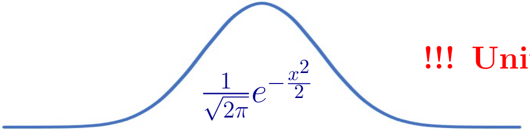




**!!! Universalité !!!**

**Thm (TCL):** $X_1, X_2 \ldots$ sont i.i.d. avec $\mathbb{E}[X_i] = 0$ et $\mathrm{Var}(X_i) = \sigma^2$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow[n \to \infty]{(d)} \mathcal{N}(0, \sigma^2)$$

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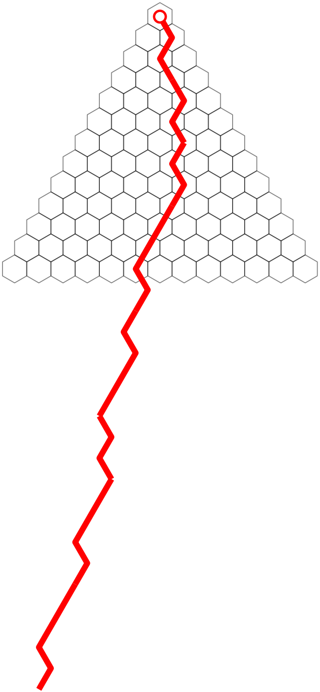
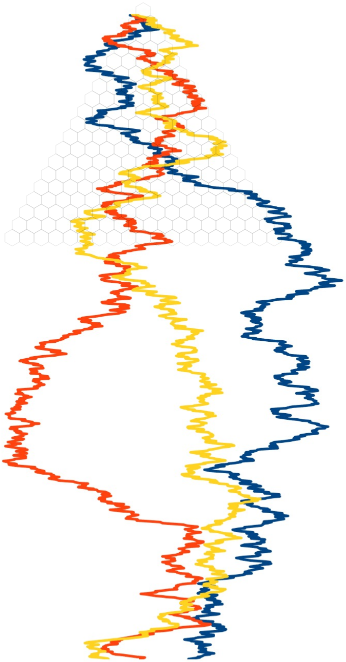
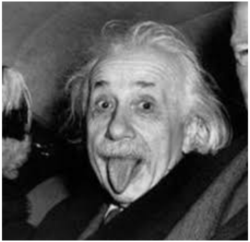
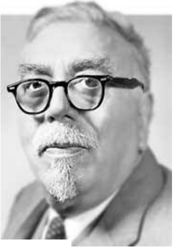
$$\mathbb{P}\Big[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \in [a, b]\Big] \xrightarrow[n\to\infty]{} \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx$$

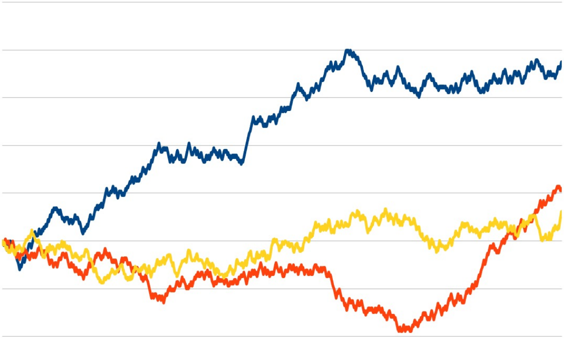




# Mouvement brownien





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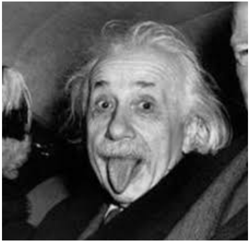


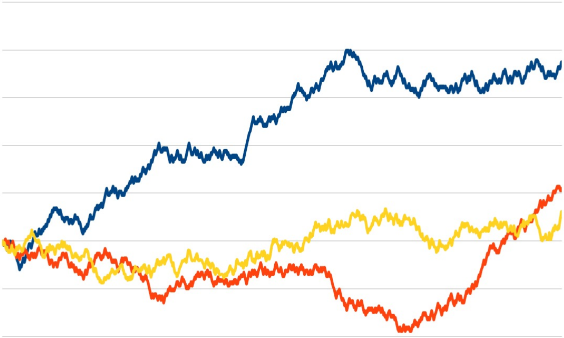

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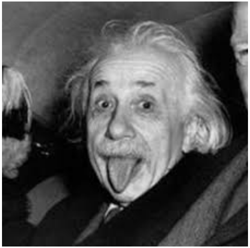

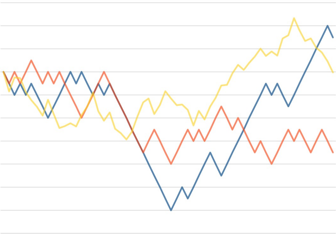

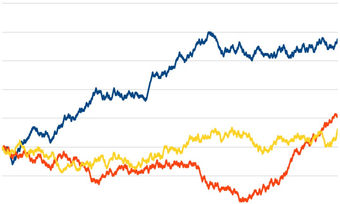

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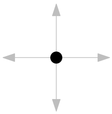
**Thm (Donsker):** *"Toute marche aléatoire (de pas d'espérance nulle et variance finie) converge vers le Mouvement Brownien."*

# Marche aléatoire en dimension $d \geq 2$



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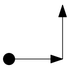

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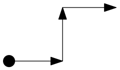

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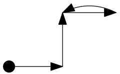

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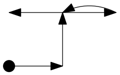

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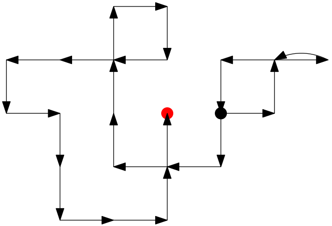

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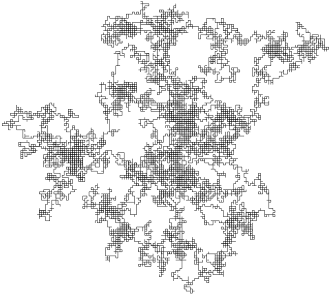

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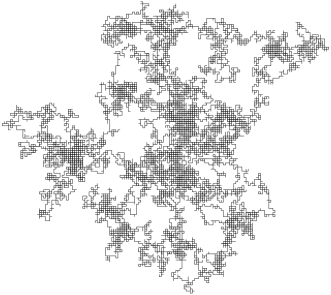

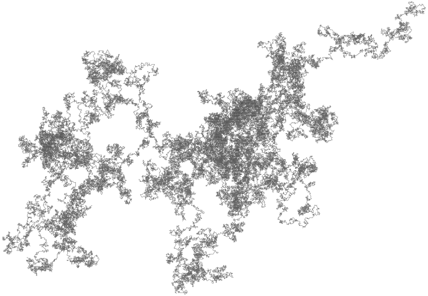

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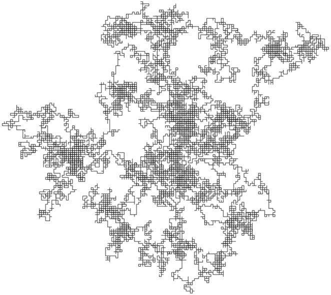

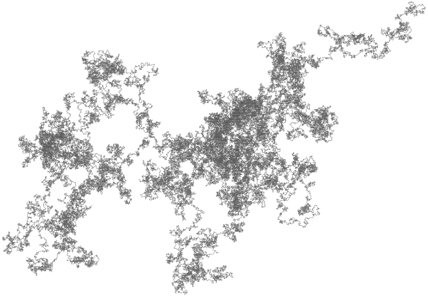

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**Question:** *La marche aléatoire sur* $\mathbb{Z}^d$, *retourne-elle en* 0 **surement?**

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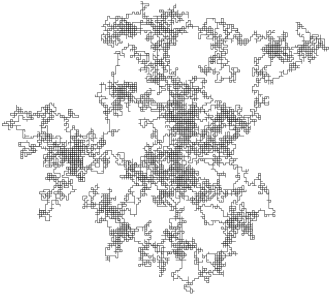

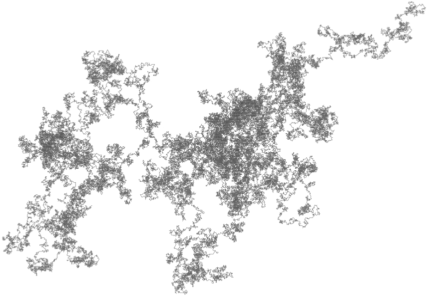

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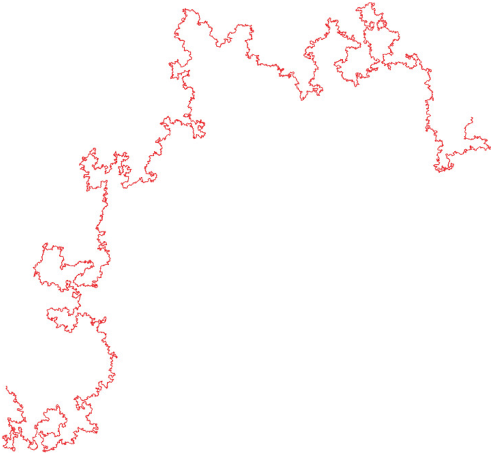
**Question:** *La marche aléatoire sur* $\mathbb{Z}^d$, *retourne-elle en* $0$ **un nombre infini de fois?**

# Marches auto-évitantes



**Marches auto-évitantes:** modèle de polymers (Flory, Orr 1947)

Marche sur $\mathbb{Z}^d$ **qui ne passe pas deux fois par le même point**.

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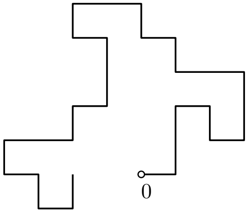
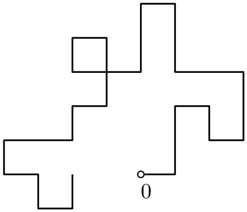



auto-évitante




pas auto-évitante
(marche simple)

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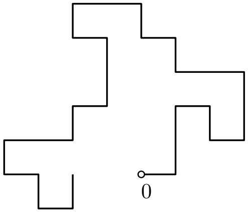


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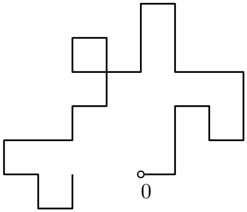


pas auto-évitante
(marche simple)

| | | |
|---|---|---|
| **Nombre marches** $n$ **pas:** | $c_n \approx \mu_c(d)^n$ | $(2d)^n$ |

# Comptage des marches auto-évitantes sur $\mathbb{Z}^2$

| $n$ = # pas | $2^n$ | $c_n$ = # marches auto-évitantes | $4^n$ nombre de marches |
|---|---|---|---|
| 2 | 4 | 12 | 16 |
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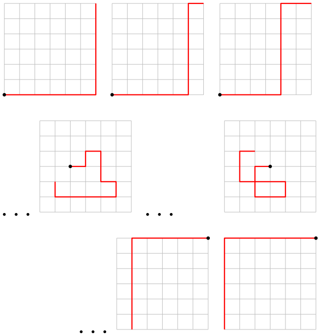
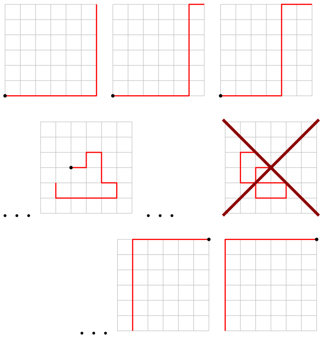
| 70 | 1180591620717410000000 | 1580784678250571882017480243636 | 1393796574908160000000000000000000000000000 |

Marches simples: $4^n$



Marches auto-évitantes: $c_n$



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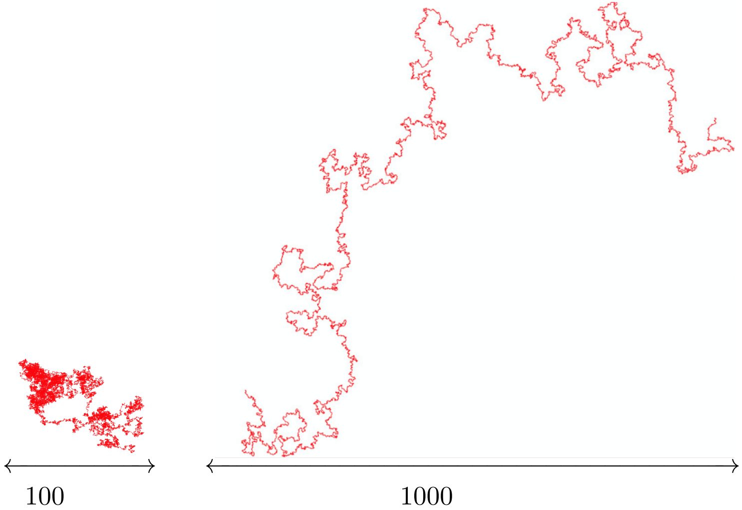
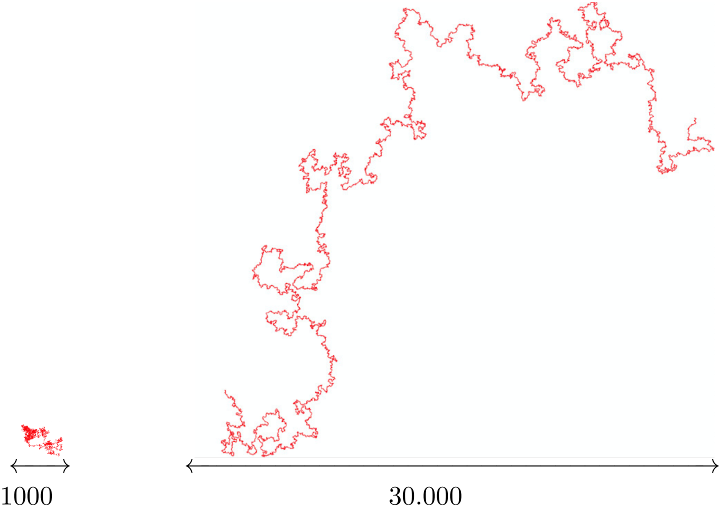
$$\mu_c = \lim_{n\to\infty} c_n^{1/n} \approx 2.63815853(15).$$

**Marche simple v auto-évitante**; même échelle (n = 10.000 pas)




**Diamèter typique:** $\sqrt{n}$ $n^{3/4}$

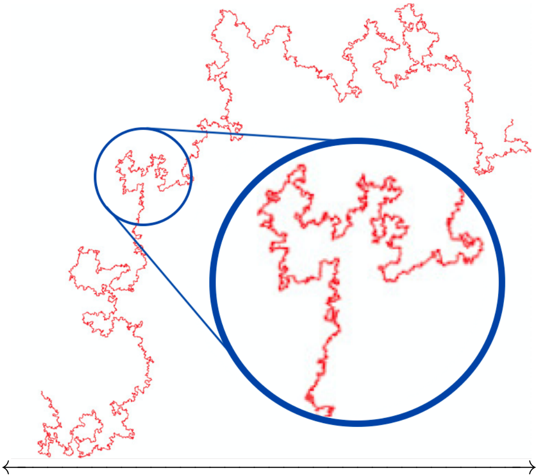
**Marche simple v auto-évitante**; même échelle (n = 1.000.000 pas)




**Diamèter typique:** $\sqrt{n}$ $n^{3/4}$

**Marche simple v auto-évitante**; même échelle (n = 1.000.000 pas)

- Invariance par changement d'échelle
- Courbe fractale
- Dim. Hausdorff> 1
- **Universalité!**



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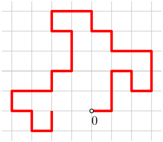


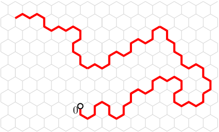


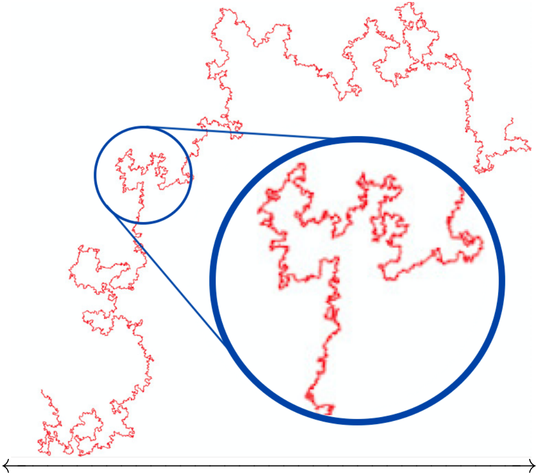

**Marche simple v auto-évitante**; même échelle (n = 1.000.000 pas)

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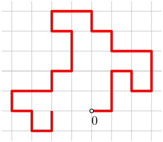
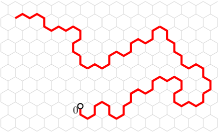
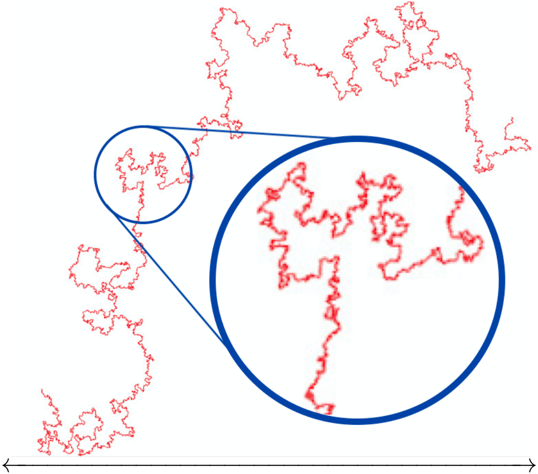
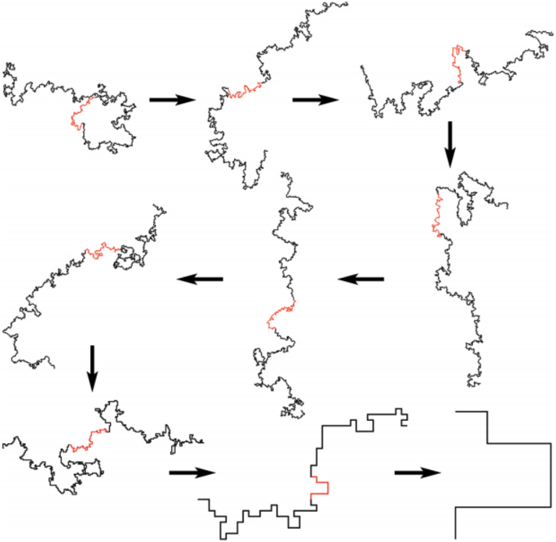


**Conjecture:** Pour $d = 2$, $\quad n^{1/2+\epsilon} < \|\Gamma_n\| < n^{1-\epsilon}$.

# Marche auto-évitante: zoom



Notices of the American Mathematical Society April 2021

Merci pour votre attention!